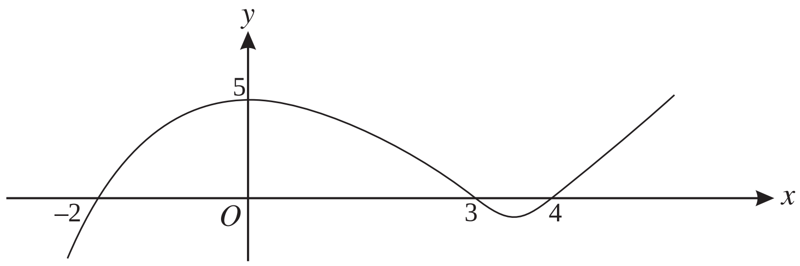
1 It is given that $f(x) = \dfrac{2ax}{(x-2a)(x^2+a^2)}$, where a is a non-zero constant. Express $f(x)$ in partial fractions. [5]

2



The diagram shows the curve $y = f(x)$. The curve has a maximum point at (0, 5) and crosses the x-axis at (−2, 0), (3, 0) and (4, 0). Sketch the curve $y^2 = f(x)$, showing clearly the coordinates of any turning points and of any points where this curve crosses the axes. [5]

3 By using the substitution $t = \tan \frac{1}{2}x$, find the exact value of

$$\int_0^{\frac{1}{2}\pi} \frac{1}{2-\cos x}\,dx,$$

giving the answer in terms of π. [6]

4 **(i)** Sketch, on the same diagram, the curves with equations $y = \operatorname{sech} x$ and $y = x^2$. [3]

(ii) By using the definition of $\operatorname{sech} x$ in terms of e^x and e^{-x}, show that the x-coordinates of the points at which these curves meet are solutions of the equation

$$x^2 = \frac{2e^x}{e^{2x}+1}.$$ [3]

(iii) The iteration

$$x_{n+1} = \sqrt{\frac{2e^{x_n}}{e^{2x_n}+1}}$$

can be used to find the positive root of the equation in part **(ii)**. With initial value $x_1 = 1$, the approximations $x_2 = 0.8050$, $x_3 = 0.8633$, $x_4 = 0.8463$ and $x_5 = 0.8513$ are obtained, correct to 4 decimal places. State with a reason whether, in this case, the iteration produces a 'staircase' or a 'cobweb' diagram. [2]

5 It is given that, for $n \geqslant 0$,

$$I_n = \int_0^{\frac{1}{4}\pi} \tan^n x\,dx.$$

(i) By considering $I_n + I_{n-2}$, or otherwise, show that, for $n \geqslant 2$,

$$(n-1)(I_n + I_{n-2}) = 1.$$ [4]

(ii) Find I_4 in terms of π. [4]

6 It is given that $f(x) = 1 - \dfrac{7}{x^2}$.

(i) Use the Newton-Raphson method, with a first approximation $x_1 = 2.5$, to find the next approximations x_2 and x_3 to a root of $f(x) = 0$. Give the answers correct to 6 decimal places. [3]

(ii) The root of $f(x) = 0$ for which x_1, x_2 and x_3 are approximations is denoted by α. Write down the exact value of α. [1]

(iii) The error e_n is defined by $e_n = \alpha - x_n$. Find e_1, e_2 and e_3, giving your answers correct to 5 decimal places. Verify that $e_3 \approx \dfrac{e_2^3}{e_1^2}$. [3]

7 It is given that $f(x) = \tanh^{-1}\left(\dfrac{1-x}{2+x}\right)$, for $x > -\frac{1}{2}$.

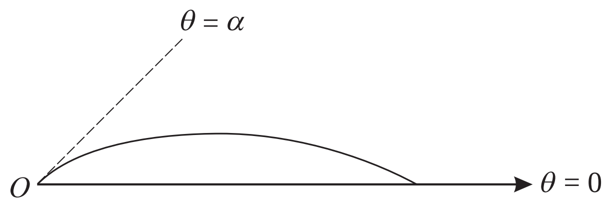
(i) Show that $f'(x) = -\dfrac{1}{1+2x}$, and find $f''(x)$. [6]

(ii) Show that the first three terms of the Maclaurin series for $f(x)$ can be written as $\ln a + bx + cx^2$, for constants a, b and c to be found. [4]

8 The equation of a curve, in polar coordinates, is

$$r = 1 - \sin 2\theta, \quad \text{for } 0 \leqslant \theta < 2\pi.$$

(i)



The diagram shows the part of the curve for which $0 \leqslant \theta \leqslant \alpha$, where $\theta = \alpha$ is the equation of the tangent to the curve at O. Find α in terms of π. [2]

(ii) **(a)** If $f(\theta) = 1 - \sin 2\theta$, show that $f\left(\frac{1}{2}(2k+1)\pi - \theta\right) = f(\theta)$ for all θ, where k is an integer. [3]

(b) Hence state the equations of the lines of symmetry of the curve

$$r = 1 - \sin 2\theta, \quad \text{for } 0 \leqslant \theta < 2\pi.$$ [2]

(iii) Sketch the curve with equation

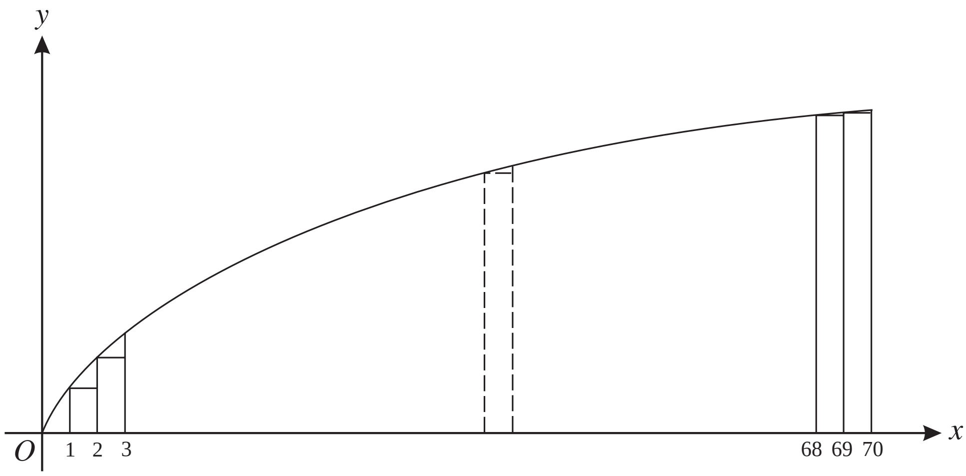
$$r = 1 - \sin 2\theta, \quad \text{for } 0 \leqslant \theta < 2\pi.$$

State the maximum value of r and the corresponding values of θ. [4]

[Turn over

9 **(i)** Prove that $\int_0^N \ln(1+x)\,dx = (N+1)\ln(N+1) - N$, where N is a positive constant. [4]

(ii)



The diagram shows the curve $y = \ln(1+x)$, for $0 \leqslant x \leqslant 70$, together with a set of rectangles of unit width.

(a) By considering the areas of these rectangles, explain why

$$\ln 2 + \ln 3 + \ln 4 + \ldots + \ln 70 < \int_0^{70} \ln(1+x)\,dx.$$ [2]

(b) By considering the areas of another set of rectangles, show that

$$\ln 2 + \ln 3 + \ln 4 + \ldots + \ln 70 > \int_0^{69} \ln(1+x)\,dx.$$ [3]

(c) Hence find bounds between which $\ln(70!)$ lies. Give the answers correct to 1 decimal place. [3]

1 It is given that $f(x) = x^2 - \sin x$.

(i) The iteration $x_{n+1} = \sqrt{\sin x_n}$, with $x_1 = 0.875$, is to be used to find a real root, α, of the equation $f(x) = 0$. Find x_2, x_3 and x_4, giving the answers correct to 6 decimal places. **[2]**

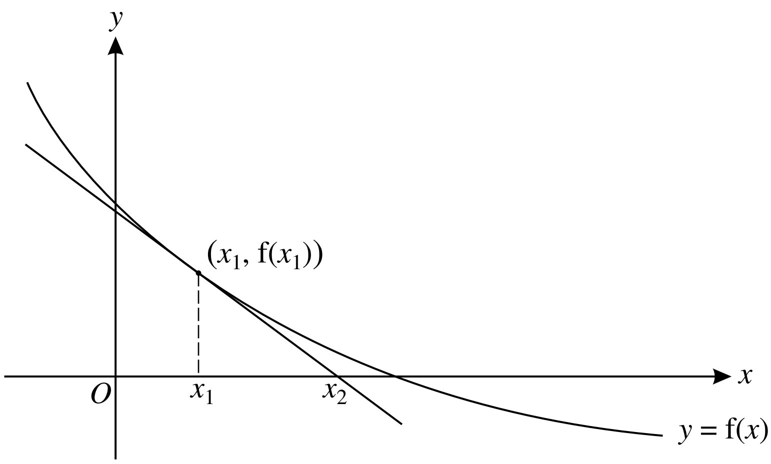
(ii) The error e_n is defined by $e_n = \alpha - x_n$. Given that $\alpha = 0.876\,726$, correct to 6 decimal places, find e_3 and e_4. Given that $g(x) = \sqrt{\sin x}$, use e_3 and e_4 to estimate $g'(\alpha)$. **[3]**

2 It is given that $f(x) = \tan^{-1}(1 + x)$.

(i) Find $f(0)$ and $f'(0)$, and show that $f''(0) = -\frac{1}{2}$. **[4]**

(ii) Hence find the Maclaurin series for $f(x)$ up to and including the term in x^2. **[2]**

3



A curve with no stationary points has equation $y = f(x)$. The equation $f(x) = 0$ has one real root α, and the Newton-Raphson method is to be used to find α. The tangent to the curve at the point $(x_1, f(x_1))$ meets the x-axis where $x = x_2$ (see diagram).

(i) Show that $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$. **[3]**

(ii) Describe briefly, with the help of a sketch, how the Newton-Raphson method, using an initial approximation $x = x_1$, gives a sequence of approximations approaching α. **[2]**

(iii) Use the Newton-Raphson method, with a first approximation of 1, to find a second approximation to the root of $x^2 - 2\sinh x + 2 = 0$. **[2]**

4 The equation of a curve, in polar coordinates, is

$$r = e^{-2\theta}, \quad \text{for } 0 \leqslant \theta \leqslant \pi.$$

(i) Sketch the curve, stating the polar coordinates of the point at which r takes its greatest value. **[2]**

(ii) The pole is O and points P and Q, with polar coordinates (r_1, θ_1) and (r_2, θ_2) respectively, lie on the curve. Given that $\theta_2 > \theta_1$, show that the area of the region enclosed by the curve and the lines OP and OQ can be expressed as $k(r_1^2 - r_2^2)$, where k is a constant to be found. **[5]**

5 **(i)** Using the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x}, show that

$$\cosh^2 x - \sinh^2 x \equiv 1.$$

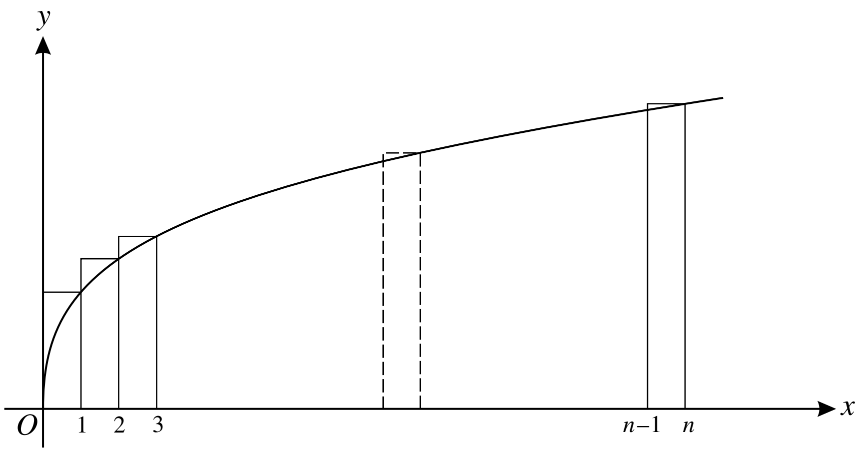
Deduce that $1 - \tanh^2 x \equiv \text{sech}^2 x$. **[4]**

(ii) Solve the equation $2\tanh^2 x - \text{sech}\, x = 1$, giving your answer(s) in logarithmic form. **[4]**

6 **(i)** Express $\dfrac{4}{(1-x)(1+x)(1+x^2)}$ in partial fractions. **[5]**

(ii) Show that $\displaystyle\int_0^{\frac{1}{\sqrt{3}}} \frac{4}{1-x^4}\,dx = \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) + \tfrac{1}{3}\pi$. **[4]**

7



The diagram shows the curve with equation $y = \sqrt[3]{x}$, together with a set of n rectangles of unit width.

(i) By considering the areas of these rectangles, explain why

$$\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \dots + \sqrt[3]{n} > \int_0^n \sqrt[3]{x}\,dx.$$ **[2]**

(ii) By drawing another set of rectangles and considering their areas, show that

$$\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \dots + \sqrt[3]{n} < \int_1^{n+1} \sqrt[3]{x}\,dx.$$ **[3]**

(iii) Hence find an approximation to $\displaystyle\sum_{n=1}^{100} \sqrt[3]{n}$, giving your answer correct to 2 significant figures. **[3]**

[Questions 8 and 9 are printed overleaf.]

Turn over

8 The equation of a curve is

$$y = \frac{kx}{(x-1)^2},$$

where k is a positive constant.

(i) Write down the equations of the asymptotes of the curve. **[2]**

(ii) Show that $y \geqslant -\frac{1}{4}k$. **[4]**

(iii) Show that the x-coordinate of the stationary point of the curve is independent of k, and sketch the curve. **[4]**

9 **(i)** Given that $y = \tanh^{-1} x$, for $-1 < x < 1$, prove that $y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$. **[3]**

(ii) It is given that $f(x) = a\cosh x - b\sinh x$, where a and b are positive constants.

(a) Given that $b \geqslant a$, show that the curve with equation $y = f(x)$ has no stationary points. **[3]**

(b) In the case where $a > 1$ and $b = 1$, show that $f(x)$ has a minimum value of $\sqrt{a^2 - 1}$. **[6]**

1 Express $\dfrac{2x+3}{(x+3)(x^2+9)}$ in partial fractions. **[5]**

2 A curve has equation $y = \dfrac{x^2 - 6x - 5}{x - 2}$.

(i) Find the equations of the asymptotes. **[3]**

(ii) Show that y can take all real values. **[4]**

3 It is given that $\mathrm{F}(x) = 2 + \ln x$. The iteration $x_{n+1} = \mathrm{F}(x_n)$ is to be used to find a root, α, of the equation $x = 2 + \ln x$.

(i) Taking $x_1 = 3.1$, find x_2 and x_3, giving your answers correct to 5 decimal places. **[2]**

(ii) The error e_n is defined by $e_n = \alpha - x_n$. Given that $\alpha = 3.146\,19$, correct to 5 decimal places, use the values of e_2 and e_3 to make an estimate of $\mathrm{F}'(\alpha)$ correct to 3 decimal places. State the true value of $\mathrm{F}'(\alpha)$ correct to 4 decimal places. **[3]**

(iii) Illustrate the iteration by drawing a sketch of $y = x$ and $y = \mathrm{F}(x)$, showing how the values of x_n approach α. State whether the convergence is of the 'staircase' or 'cobweb' type. **[3]**

4 A curve C has the cartesian equation $x^3 + y^3 = axy$, where $x \geqslant 0$, $y \geqslant 0$ and $a > 0$.

(i) Express the polar equation of C in the form $r = \mathrm{f}(\theta)$ and state the limits between which θ lies. **[3]**

The line $\theta = \alpha$ is a line of symmetry of C.

(ii) Find and simplify an expression for $\mathrm{f}\left(\frac{1}{2}\pi - \theta\right)$ and hence explain why $\alpha = \frac{1}{4}\pi$. **[3]**

(iii) Find the value of r when $\theta = \frac{1}{4}\pi$. **[1]**

(iv) Sketch the curve C. **[2]**

5 **(i)** Prove that, if $y = \sin^{-1} x$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\sqrt{1 - x^2}}$. **[3]**

(ii) Find the Maclaurin series for $\sin^{-1} x$, up to and including the term in x^3. **[5]**

(iii) Use the result of part **(ii)** and the Maclaurin series for $\ln(1 + x)$ to find the Maclaurin series for $(\sin^{-1} x)\ln(1 + x)$, up to and including the term in x^4. **[4]**

6 It is given that $I_n = \displaystyle\int_0^1 x^n (1 - x)^{\frac{3}{2}}\,\mathrm{d}x$, for $n \geqslant 0$.

(i) Show that $I_n = \dfrac{2n}{2n + 5} I_{n-1}$, for $n \geqslant 1$. **[6]**

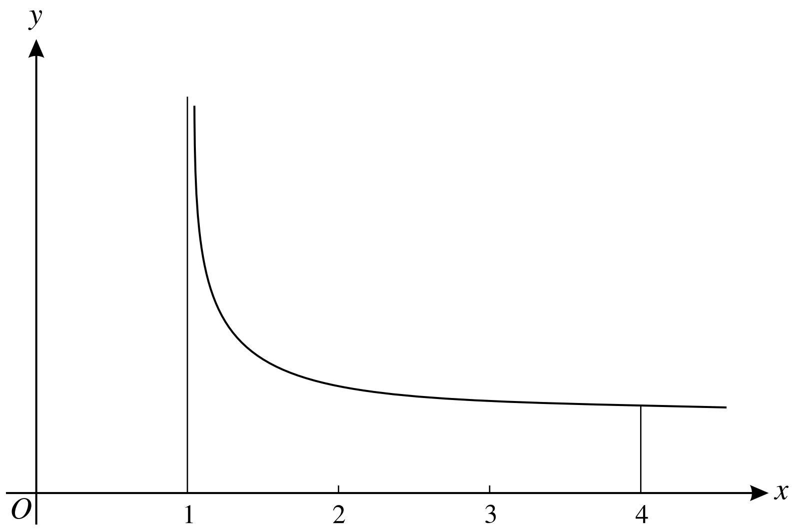
(ii) Hence find the exact value of I_3. **[4]**

7 **(i)** Sketch the graph of $y = \tanh x$ and state the value of the gradient when $x = 0$. On the same axes, sketch the graph of $y = \tanh^{-1} x$. Label each curve and give the equations of the asymptotes. **[4]**

(ii) Find $\int_0^k \tanh x \, dx$, where $k > 0$. **[2]**

(iii) Deduce, or show otherwise, that $\int_0^{\tanh k} \tanh^{-1} x \, dx = k \tanh k - \ln(\cosh k)$. **[4]**

8 **(i)** Use the substitution $x = \cosh^2 u$ to find $\int \sqrt{\frac{x}{x-1}} \, dx$, giving your answer in the form $f(x) + \ln\big(g(x)\big)$. **[7]**



(ii) Hence calculate the exact area of the region between the curve $y = \sqrt{\frac{x}{x-1}}$, the x-axis and the lines $x = 1$ and $x = 4$ (see diagram). **[1]**

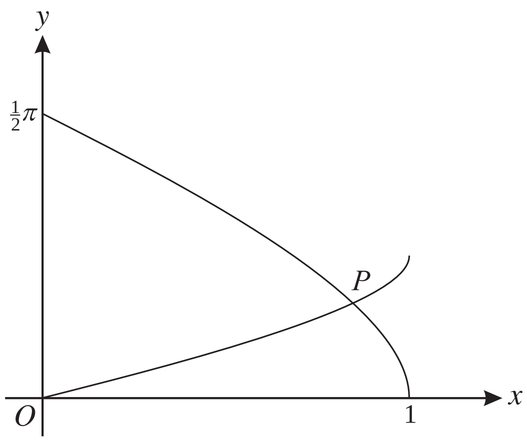
(iii) What can you say about the volume of the solid of revolution obtained when the region defined in part **(ii)** is rotated completely about the x-axis? Justify your answer. **[3]**

1 It is given that $\mathrm{f}(x) = \ln(1 + \cos x)$.

(i) Find the exact values of $\mathrm{f}(0)$, $\mathrm{f}'(0)$ and $\mathrm{f}''(0)$. [4]

(ii) Hence find the first two non-zero terms of the Maclaurin series for $\mathrm{f}(x)$. [2]

2

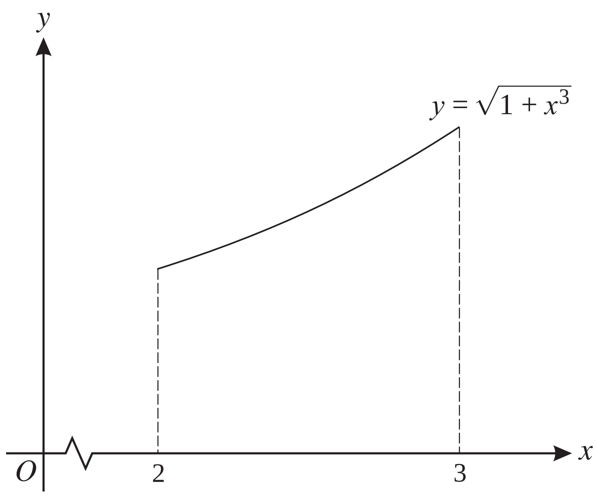


The diagram shows parts of the curves with equations $y = \cos^{-1} x$ and $y = \frac{1}{2}\sin^{-1} x$, and their point of intersection P.

(i) Verify that the coordinates of P are $\left(\frac{1}{2}\sqrt{3}, \frac{1}{6}\pi\right)$. [2]

(ii) Find the gradient of each curve at P. [3]

3



The diagram shows the curve with equation $y = \sqrt{1 + x^3}$, for $2 \leqslant x \leqslant 3$. The region under the curve between these limits has area A.

(i) Explain why $3 < A < \sqrt{28}$. [2]

(ii) The region is divided into 5 strips, each of width 0.2. By using suitable rectangles, find improved lower and upper bounds between which A lies. Give your answers correct to 3 significant figures. [4]

4 The equation of a curve, in polar coordinates, is

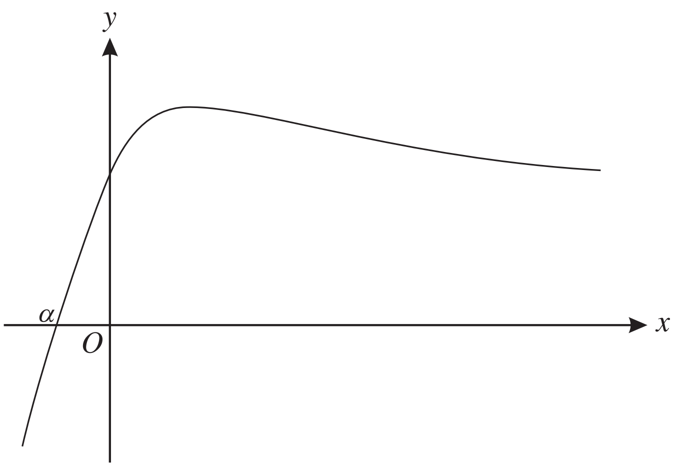
$$r = 1 + 2\sec\theta, \quad \text{for } -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi.$$

(i) Find the exact area of the region bounded by the curve and the lines $\theta = 0$ and $\theta = \frac{1}{6}\pi$. [5]

[The result $\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta|$ may be assumed.]

(ii) Show that a cartesian equation of the curve is $(x-2)\sqrt{x^2+y^2} = x$. [3]

5



The diagram shows the curve with equation $y = xe^{-x} + 1$. The curve crosses the x-axis at $x = \alpha$.

(i) Use differentiation to show that the x-coordinate of the stationary point is 1. [2]

α is to be found using the Newton-Raphson method, with $f(x) = xe^{-x} + 1$.

(ii) Explain why this method will not converge to α if an initial approximation x_1 is chosen such that $x_1 > 1$. [2]

(iii) Use this method, with a first approximation $x_1 = 0$, to find the next three approximations x_2, x_3 and x_4. Find α, correct to 3 decimal places. [5]

6 The equation of a curve is $y = \dfrac{2x^2 - 11x - 6}{x - 1}$.

(i) Find the equations of the asymptotes of the curve. [3]

(ii) Show that y takes all real values. [5]

[Turn over

7 It is given that, for integers $n \geqslant 1$,

$$I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx.$$

(i) Use integration by parts to show that $I_n = 2^{-n} + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx$. [3]

(ii) Show that $2nI_{n+1} = 2^{-n} + (2n-1)I_n$. [3]

(iii) Find I_2 in terms of π. [3]

8 **(i)** By using the definition of $\sinh x$ in terms of e^x and e^{-x}, show that

$$\sinh^3 x = \tfrac{1}{4}\sinh 3x - \tfrac{3}{4}\sinh x.$$ [4]

(ii) Find the range of values of the constant k for which the equation

$$\sinh 3x = k\sinh x$$

has real solutions other than $x = 0$. [3]

(iii) Given that $k = 4$, solve the equation in part **(ii)**, giving the non-zero answers in logarithmic form. [3]

9 **(i)** Prove that $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$. [3]

(ii) Hence, or otherwise, find $\int \frac{1}{\sqrt{4x^2-1}}\,dx$. [2]

(iii) By means of a suitable substitution, find $\int \sqrt{4x^2-1}\,dx$. [6]