1 Andy (A), Beth (B), Chelsey (C), Dean (D) and Elly (E) have formed a quiz team. They have entered a quiz in which, as well as team questions, each of them must answer individual questions on a specialist topic. The specialist topics could be any of: food (F), geography (G), history (H), politics (P), science (S) and television (T). The team members do not know which five specialist topics will arise in the quiz.

Andy wants to answer questions on either food or television; Beth wants to answer questions on geography, history or science; Chelsey wants to answer questions on geography or television; Dean wants to answer questions on politics or television; and Elly wants to answer questions on history or television.

(i) Draw a bipartite graph to show the possible pairings between the team members and the specialist topics. **[1]**

In the quiz, the first specialist topic is food, and Andy is chosen to answer the questions. The second specialist topic is geography, and Beth is chosen. The next specialist topic is history, and Elly is chosen. The fourth specialist topic is science. Beth has already answered questions so Dean offers to try this round. The final specialist topic is television, and Chelsey answers these questions.

(ii) Draw a second bipartite graph to show these pairings, apart from Dean answering the science questions. Write down an alternating path starting from Dean to show that there would have been a better way to choose who answered the questions had the topics been known in advance. Write down which team member would have been chosen for each specialist topic in this case. **[4]**

(iii) In a practice, although the other team members were able to choose topics that they wanted, Beth had to answer the questions on television. Write down which topic each team member answered questions on, and which topic did not arise. **[2]**

2 Dudley has three daughters who are all planning to get married next year. The girls are named April, May and June, after the months in which they were born. Each girl wants to get married on her own birthday.

Dudley has already obtained costings from four different hotels. From past experience, Dudley knows that when his family get together they are likely to end up with everyone fighting one another, so he cannot use the same hotel twice.

The table shows the costs, in £100, for each hotel to host each daughter's wedding.

		Hotel			
		Palace	Regent	Sunnyside	Tall Trees
Daughter	April	30	28	32	25
	May	32	34	32	35
	June	40	40	39	38

Dudley wants to choose the three hotels to minimise the total cost.

Add a dummy row and then apply the Hungarian algorithm to the table, reducing rows first, to find an optimal allocation between the hotels and Dudley's daughters. State how each table is formed and write out the final solution and its cost to Dudley. **[7]**

3 The table lists the duration (in hours), immediate predecessors and number of workers required for each activity in a project.

Activity	Duration	Immediate predecessors	Number of workers
A	6	–	2
B	5	–	4
C	4	–	1
D	1	A, B	3
E	2	B	2
F	1	B, C	2
G	2	D, E	4
H	3	D, E, F	3

(i) Draw an activity network, using activity on arc, to represent the project. You should make your diagram quite large so that there is room for working. **[2]**

(ii) Carry out a forward pass and a backward pass through the activity network, showing the early and late event times clearly at the vertices of your network.

State the minimum project completion time and list the critical activities. **[6]**

(iii) Using graph paper, draw a resource histogram to show the number of workers required each hour. Each activity begins at its earliest possible start time. Once an activity has started it runs for its duration without a break. **[2]**

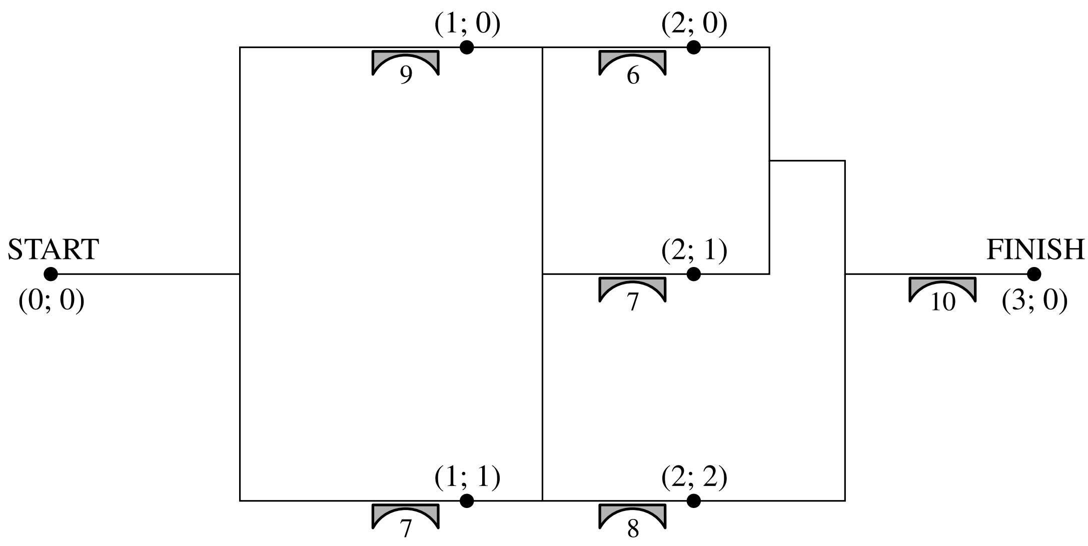
A delay from the supplier means that the start of activity F is delayed.

(iv) By how much could the start of activity F be delayed without affecting the minimum project completion time? **[1]**

Suppose that only six workers are available after the first four hours of the project.

(v) Explain carefully what delay this will cause on the completion of the project. What is the maximum possible delay on the start of activity F, compared with its earliest possible start time in part **(iii)**, without affecting the new minimum project completion time? Justify your answer. **[4]**

4 The diagram represents a map of an army truck-driving course that includes several bridges. The start and a 'safe point' just after each bridge have been given (stage; state) labels. The number below each bridge shows its weight limit, in tonnes.



An army cadet needs to drive a truck through the course from start to finish, crossing exactly three bridges.

(i) Draw a network, using the (stage; state) labels given, to represent the routes through the course. The weights on the arcs should show the weight limits for the bridges. **[3]**

The cadet wants to find out the weight of the heaviest truck she can use.

(ii) Which network problem does she need to solve? **[1]**

(iii) Set up a dynamic programming tabulation to solve the cadet's problem. Write down the weight of the heaviest truck she can use and write down the (stage; state) labels for the route she should take. **[9]**

5 Robbie received a new computer game for Christmas. He has already worked through several levels of the game but is now stuck at one of the levels in which he is playing against a character called Conan. Robbie has played this particular level several times.

Each time Robbie encounters Conan he can choose to be helped by a dwarf, an elf or a fairy. Conan chooses between being helped by a goblin, a hag or an imp. The players make their choices simultaneously, without knowing what the other has chosen.

Robbie starts the level with ten gold coins. The table shows the number of gold coins that Conan must give Robbie in each encounter for each combination of helpers (a negative entry means that Robbie gives gold coins to Conan). If Robbie's total reaches twenty gold coins then he completes the level, but if it reaches zero the game ends. This means that each attempt can be regarded as a zero-sum game.

		Conan		
		Goblin	Hag	Imp
Robbie	Dwarf	–1	–4	2
	Elf	3	1	–4
	Fairy	1	–1	1

(i) Find the play-safe choice for each player, showing your working. Which helper should Robbie choose if he thinks that Conan will play-safe? **[5]**

(ii) How many gold coins can Robbie expect to win, with each choice of helper, if he thinks that Conan will choose randomly between his three choices (so that each has probability $\frac{1}{3}$)? **[2]**

Robbie decides to choose his helper by using random numbers to choose between the elf and the fairy, where the probability of choosing the elf is p and the probability of choosing the fairy is $1 - p$.

(iii) Write down an expression for the expected number of gold coins won at each encounter by Robbie for each of Conan's choices. Calculate the optimal value of p. **[4]**

Robbie's girlfriend thinks that he should have included the possibility of choosing the dwarf. She denotes the probability with which Robbie should choose the dwarf, the elf and the fairy as x, y and z respectively. She then models the problem of choosing between the three helpers as the following LP.

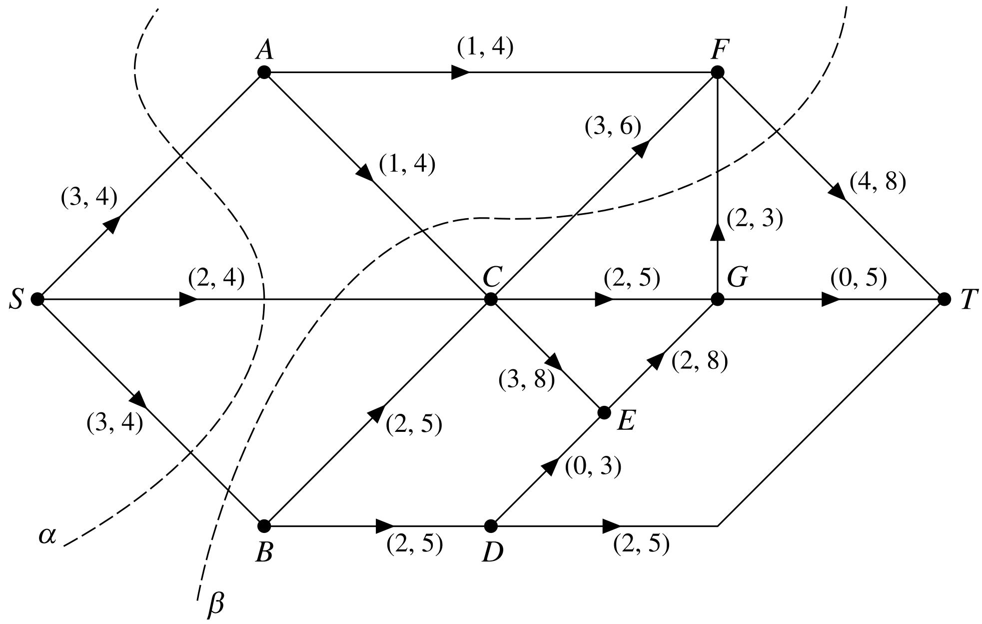
$$\begin{array}{ll} \text{Maximise} & M = m - 4, \\ \text{subject to} & m \leqslant 3x + 7y + 5z, \\ & m \leqslant 5y + 3z, \\ & m \leqslant 6x + 5z, \\ & x + y + z \leqslant 1, \\ \text{and} & m \geqslant 0,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0. \end{array}$$

(iv) Explain how the expression $3x + 7y + 5z$ was formed. **[2]**

Robbie's girlfriend uses the Simplex algorithm to solve the LP problem. Her solution has $x = 0$ and $y = \frac{2}{7}$.

(v) Calculate the optimal value of M. **[3]**

6 The diagram represents a system of pipes through which fluid can flow from a source, S, to a sink, T. It also shows two cuts, α and β. The weights on the arcs show the lower and upper capacities of the pipes in litres per second.



(i) Calculate the capacities of the cuts α and β. **[2]**

(ii) Explain why the arcs AC and AF cannot both be at their lower capacities. **[1]**

(iii) Explain why the arcs BC, BD, DE and DT must all be at their lower capacities. **[2]**

(iv) Show that a flow of 10 litres per second is impossible. Deduce the minimum and maximum feasible flows, showing your working. **[6]**

Vertex E becomes blocked so that no fluid can flow through it.

(v) Draw a copy of the network with this vertex restriction. You are advised to make your diagram quite large. Show a flow of 9 litres per second on your diagram. **[3]**

1 **(a)** A café sells five different types of filled roll. Mr King buys one of each to take home to his family. The family consists of Mr King's daughter Fiona (F), his mother Gwen (G), his wife Helen (H), his son Jack (J) and Mr King (K).

The table shows who likes which rolls.

		F	G	H	J	K
Avocado and bacon	(A)	✓		✓		
Beef and horseradish	(B)		✓	✓	✓	✓
Chicken and stuffing	(C)		✓		✓	
Duck and plum sauce	(D)			✓		✓
Egg and tomato	(E)	✓				

(i) Draw a bipartite graph to represent this information. Put the fillings (A, B, C, D and E) on the left-hand side and the members of the family (F, G, H, J and K) on the right-hand side. **[1]**

Fiona takes the avocado roll; Gwen takes the beef roll; Helen takes the duck roll and Jack takes the chicken roll.

(ii) Draw a second bipartite graph to show this incomplete matching. **[1]**

(iii) Construct the shortest possible alternating path from E to K and hence find a complete matching. State which roll each family member has with this complete matching. **[2]**

(iv) Find a different complete matching. **[1]**

(b) Mr King decides that the family should eat more fruit. Each family member gives a score out of 10 to five fruits. These scores are subtracted from 10 to give the values below.

Family member

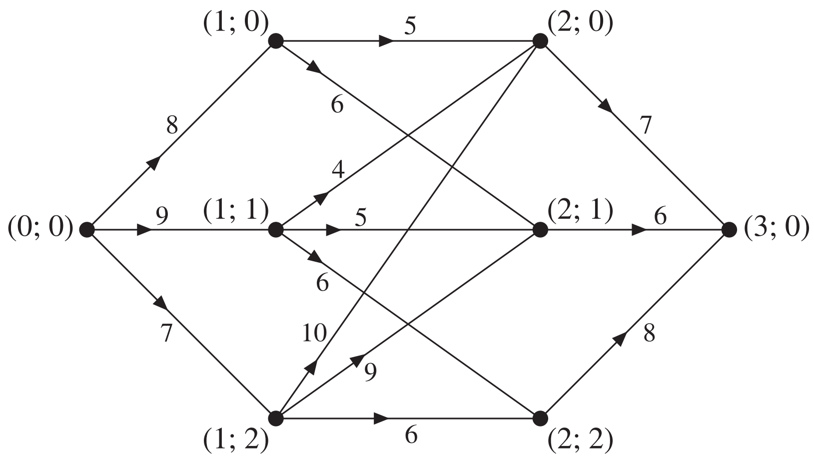
		F	G	H	J	K
Lemon	L	8	8	8	8	1
Mandarin	M	4	8	6	4	2
Nectarine	N	9	9	9	7	1
Orange	O	4	6	5	4	3
Peach	P	6	9	7	5	0

The smaller entries in each column correspond to fruits that the family members liked most.

Mr King buys one of each of these five fruits. Each family member is to be given a fruit.

Apply the Hungarian algorithm, **reducing rows first**, to find a minimum cost matching. You must show your working clearly. Which family member should be given which fruit? **[8]**

2 **(i)** Set up a dynamic programming tabulation to find the maximum weight route from (0; 0) to (3; 0) on the following directed network.



Give the route and its total weight. **[11]**

(ii) The actions now represent the activities in a project and the weights represent their durations. This information is shown in the table below.

Activity	Duration	Immediate predecessors
A	8	–
B	9	–
C	7	–
D	5	A
E	6	A
F	4	B
G	5	B
H	6	B
I	10	C
J	9	C
K	6	C
L	7	D, F, I
M	6	E, G, J
N	8	H, K

Make a large copy of the network with the activities A to N labelled appropriately. Carry out a forward pass to find the early event times and a backward pass to find the late event times. Find the minimum completion time for the project and list the critical activities. **[7]**

(iii) Compare the solutions to parts **(i)** and **(ii)**. **[2]**

3 The 'Rovers' and the 'Collies' are two teams of dog owners who compete in weekly dog shows. The top three dogs owned by members of the Rovers are Prince, Queenie and Rex. The top four dogs owned by the Collies are Woof, Xena, Yappie and Zulu.

In a show the Rovers choose one of their dogs to compete against one of the dogs owned by the Collies. There are 10 points available in total. Each of the 10 points is awarded either to the dog owned by the Rovers or to the dog owned by the Collies. There are no tied points. At the end of the competition, 5 points are subtracted from the number of points won by each dog to give the score for that dog.

The table shows the score for the dog owned by the Rovers for each combination of dogs.

		Collies			
		W	X	Y	Z
	P	1	2	−1	3
Rovers	Q	−2	1	−3	−1
	R	2	−4	1	0

(i) Explain why calculating the score by subtracting 5 from the number of points for each dog makes this a zero-sum game. **[2]**

(ii) If the Rovers choose Prince and the Collies choose Woof, what score does Woof get, and how many points do Prince and Woof each get in the competition? **[2]**

(iii) Show that column W is dominated by one of the other columns, and state which column this is. **[2]**

(iv) Delete the column for W and find the play-safe strategy for the Rovers and the play-safe strategy for the Collies on the table that remains. **[3]**

Queenie is ill one week, so the Rovers make a random choice between Prince and Rex, choosing Prince with probability p and Rex with probability $1 - p$.

(v) Write down and simplify an expression for the expected score for the Rovers when the Collies choose Xena. Write down and simplify the corresponding expressions for when the Collies choose Yappie and for when they choose Zulu. **[2]**

(vi) Using graph paper, draw a graph to show the expected score for the Rovers against p for each of the choices that the Collies can make. Using your graph, find the optimal value of p for the Rovers. **[3]**

[This question continues on the next page.]

If Queenie had not been ill, the Rovers would have made a random choice between Prince, Queenie and Rex, choosing Prince with probability p_1, Queenie with probability p_2 and Rex with probability p_3.

The problem of choosing the optimal values of p_1, p_2 and p_3 can be formulated as the following linear programming problem:

$$\begin{array}{ll} \text{maximise} & M = m - 4 \\ \text{subject to} & m \leqslant 6p_1 + 5p_2, \\ & m \leqslant 3p_1 + p_2 + 5p_3, \\ & m \leqslant 7p_1 + 3p_2 + 4p_3, \\ & p_1 + p_2 + p_3 \leqslant 1 \\ \text{and} & p_1 \geqslant 0,\ p_2 \geqslant 0,\ p_3 \geqslant 0,\ m \geqslant 0. \end{array}$$

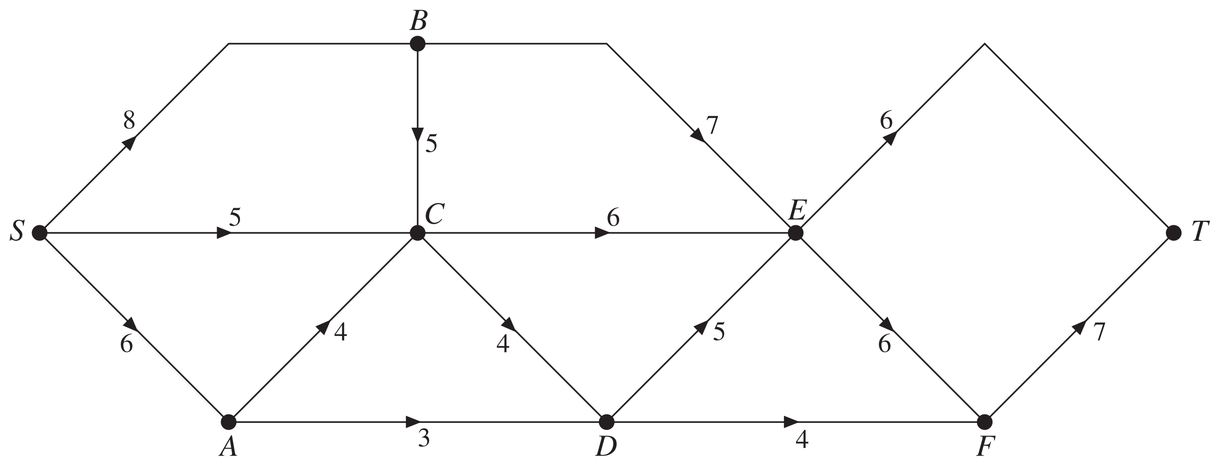
(vii) Explain how the expressions $6p_1 + 5p_2$, $3p_1 + p_2 + 5p_3$ and $7p_1 + 3p_2 + 4p_3$ were obtained. Also explain how the linear programming formulation tells you that M is a maximin solution. **[3]**

The Simplex algorithm is used to find the optimal values of the probabilities. The optimal value of p_1 is $\frac{5}{8}$ and the optimal value of p_2 is 0.

(viii) Calculate the optimal value of p_3 and the corresponding value of M. **[2]**

Turn over

4 The network represents a system of pipes through which fluid can flow from a source, S, to a sink, T. The weights on the arcs represent pipe capacities in gallons per minute.



(i) Calculate the capacity of the cut that separates $\{S, A, C, D\}$ from $\{B, E, F, T\}$. **[2]**

(ii) Explain why the arcs AC and AD cannot both be full to capacity and why the arcs DF and EF cannot both be full to capacity. **[2]**

(iii) Draw a diagram to show a flow in which as much as possible flows through vertex E but none flows through vertex A and none flows through vertex D. State the maximum flow through vertex E. **[4]**

An engineer wants to find a flow augmenting route to improve the flow from part **(iii)**.

(iv) **(a)** Explain why there can be no flow augmenting route that passes through vertex A but not through vertex D. **[1]**

(b) Write down a flow augmenting route that passes through vertex D but not through vertex A. State the maximum by which the flow can be augmented. **[2]**

(v) Prove that the augmented flow in part **(iv)(b)** is the maximum flow. **[4]**

(vi) A vertex restriction means that the flow through E can no longer be at its maximum rate. By how much can the flow through E be reduced without reducing the maximum flow from S to T? Explain your reasoning. **[3]**

The pipe represented by the arc BE becomes blocked and cannot be used.

(vii) Draw a diagram to show that, even when the flow through E is reduced as in part **(vi)**, the same maximum flow from S to T is still possible. **[2]**

1 Four friends, Amir (A), Bex (B), Cerys (C) and Duncan (D), are visiting a bird sanctuary. They have decided that they each will sponsor a different bird. The sanctuary is looking for sponsors for a kite (K), a lark (L), a moorhen (M), a nightjar (N), and an owl (O).

Amir wants to sponsor the kite, the nightjar or the owl; Bex wants to sponsor the lark, the moorhen or the owl; Cerys wants to sponsor the kite, the lark or the owl; and Duncan wants to sponsor either the lark or the owl.

(i) Draw a bipartite graph to show which friend wants to sponsor which birds. **[1]**

Amir chooses to sponsor the kite and Bex chooses the lark. Cerys then chooses the owl and Duncan is left with no bird that he wants.

(ii) Write down the shortest possible alternating path starting from the nightjar, and hence write down one way in which all four friends could have chosen birds that they wanted to sponsor. **[2]**

(iii) List a way in which all four friends could have chosen birds they wanted to sponsor, with the owl not being chosen. **[1]**

2 Amir, Bex, Cerys and Duncan all have birthdays in January. To save money they have decided that they will each buy a present for just one of the others, so that each person buys one present and receives one present. Four slips of paper with their names on are put into a hat and each person chooses one of them. They do not tell the others whose name they have chosen and, fortunately, nobody chooses their own name.

The table shows the cost, in £, of the present that each person would buy for each of the others.

		To			
		Amir	Bex	Cerys	Duncan
From	Amir	–	15	21	19
	Bex	20	–	16	14
	Cerys	25	12	–	16
	Duncan	24	10	18	–

As it happens, the names are chosen in such a way that the total cost of the presents is minimised.

Assign the cost £25 to each of the missing entries in the table and then apply the Hungarian algorithm, reducing rows first, to find which name each person chose. **[7]**

3 The table lists the duration, immediate predecessors and number of workers required for each activity in a project.

Activity	Duration (hours)	Immediate predecessors	Number of workers
A	3	–	1
B	2	–	1
C	2	A	2
D	3	A, B	2
E	3	C	3
F	3	C, D	3
G	2	D	3
H	5	E, F	1
I	4	F, G	2

(i) Represent the project by an activity network, using activity on arc. You should make your diagram quite large so that there is room for working. **[3]**

(ii) Carry out a forward pass and a backward pass through the activity network, showing the early event times and late event times clearly at the vertices of your network.

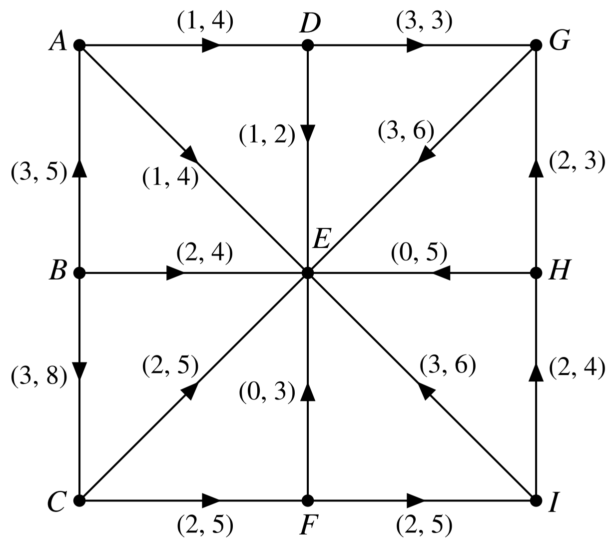
State the minimum project completion time and list the critical activities. **[5]**

(iii) Draw a resource histogram to show the number of workers required each hour when each activity begins at its earliest possible start time. **[2]**

(iv) Show how it is possible for the project to be completed in the minimum project completion time when only six workers are available. **[2]**

4 Answer parts (v) and (vi) of this question on the insert provided.

The diagram represents a system of pipes through which fluid can flow. The weights on the arcs show the lower and upper capacities of the pipes in litres per second.



(i) Which vertex is the source and which vertex is the sink? **[1]**

(ii) Cut α partitions the vertices into the sets $\{A, B, C\}$, $\{D, E, F, G, H, I\}$. Calculate the capacity of cut α. **[2]**

(iii) Explain why partitioning the vertices into sets $\{A, D, G\}$, $\{B, C, E, F, H, I\}$ does not give a cut. **[1]**

(iv) **(a)** How many litres per second must flow along arc DG? **[1]**

(b) Explain why the arc AD must be at its upper capacity. Hence find the flow in arc BA. **[2]**

(c) Explain why at least 7 litres per second must flow along arc BC. **[2]**

(v) Use the diagrams in the insert to show a minimum feasible flow and a maximum feasible flow. **[4]**

The upper capacity of BC is now increased from 8 to 18.

(vi) **(a)** Use the diagram in the insert to show a flow of 19 litres per second. **[1]**

(b) List the saturated arcs when 19 litres per second flows through the network. Hence, or otherwise, find a cut of capacity 19. **[2]**

(vii) Explain how your answers to part **(vi)** show that 19 litres per second is the maximum flow. **[2]**

5 A card game between two players consists of several rounds. In each round the players both choose a card from those in their hand; they then show these cards to each other and exchange tokens. The number of tokens that the second player gives to the first player depends on the colour of the first player's card and the design on the second player's card.

The table shows the number of tokens that the first player receives for each combination of colour and design. A negative entry means that the first player gives tokens to the second, zero means that no tokens are exchanged.

		Second player		
		Square	Triangle	Circle
	Red	2	−1	1
First player	Yellow	−2	0	−3
	Blue	−5	1	3

(i) Explain how you know that the game is zero-sum. Describe what zero-sum means for the way in which the players play the game. **[2]**

(ii) Find the play-safe choice for each player, showing your working. Explain how you know whether the game is stable or unstable. Describe what 'stable' and 'unstable' mean for the way in which the players play the game. **[6]**

The first player decides not to risk playing a blue card.

(iii) Show that in this reduced game the circle strategy dominates the square strategy, and explain what this means for the way in which the second player plays the game. **[2]**

The first player uses random numbers to choose between the other two colours, where the probability of choosing a red card is p and the probability of choosing a yellow card is $1 - p$.

(iv) Write down an expression for the expected number of tokens that the first player is given in each round for each choice of design. Calculate the optimal value of p, showing your working. **[3]**

The entries in the row for 'Blue' in the original table are now all multiplied by −1. So, for example, when the first player chooses blue and the second chooses square, instead of the first player giving the second player 5 tokens, the second player now gives the first player 5 tokens.

The first player now uses random numbers to choose between the three colours, letting x, y and z denote the probabilities of choosing red, yellow and blue respectively.

The problem of choosing between the three colours is modelled as the following LP.

Maximise $M = m - 3$,

subject to
$$m \leqslant 5x + y + 8z,$$
$$m \leqslant 2x + 3y + 2z,$$
$$m \leqslant 4x,$$
$$x + y + z \leqslant 1,$$

and $m \geqslant 0,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0.$

(v) Explain how the expression $5x + y + 8z$ was formed. **[2]**

The Simplex algorithm is used to solve the LP problem. The solution has $x = 0.6$, $y = 0.4$ and $z = 0$.

(vi) Calculate the value of each of the expressions $5x + y + 8z$, $2x + 3y + 2z$ and $4x$. Hence write down the optimal value of M. **[3]**

Turn over

6 Answer this question on the insert provided.

Four friends have decided to sponsor four birds at a bird sanctuary. They want to construct a route through the bird sanctuary, starting and ending at the entrance/exit, that enables them to visit the four birds in the shortest possible time. The table below shows the times, in minutes, that it takes to get between the different birds and the entrance/exit. The friends will spend the same amount of time with each bird, so this does not need to be included in the calculation.

	Entrance/exit	Kite	Lark	Moorhen	Nightjar
Entrance/exit	–	10	14	12	17
Kite	10	–	3	2	6
Lark	14	3	–	2	4
Moorhen	12	2	2	–	3
Nightjar	17	6	4	3	–

Let the stages be 0, 1, 2, 3, 4, 5. Stage 0 represents arriving at the sanctuary entrance. Stage 1 represents visiting the first bird, stage 2 the second bird, and so on, with stage 5 representing leaving the sanctuary. Let the states be 0, 1, 2, 3, 4 representing the entrance/exit, kite, lark, moorhen and nightjar respectively.

(i) Calculate how many minutes it takes to travel the route

$$(0; 0) – (1; 1) – (2; 2) – (3; 3) – (4; 4) – (5; 0).$$ **[1]**

The friends then realise that if they try to find the quickest route using dynamic programming with this (stage; state) formulation, they will get the route (0; 0) – (1; 1) – (2; 2) – (3; 3) – (4; 1) – (5; 0), or this in reverse, taking 27 minutes.

(ii) Explain why the route (0; 0) – (1; 1) – (2; 2) – (3; 3) – (4; 1) – (5; 0) is not a solution to the friends' problem. **[1]**

Instead, the friends set up a dynamic programming tabulation with stages and states as described above, except that now the states also show, in brackets, any birds that have already been visited. So, for example, state 1(234) means that they are currently visiting the kite and have already visited the other three birds in some order. The partially completed dynamic programming tabulation is shown opposite.

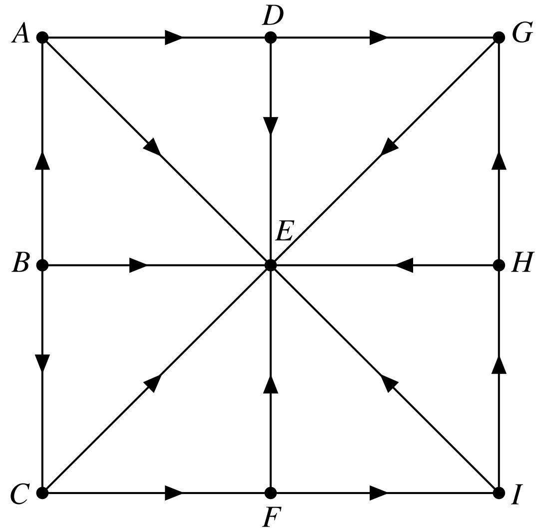
(iii) For the last completed row, i.e. stage 2, state 1(3), action 4(13), explain where the value 18 and the value 6 in the working column come from. **[2]**

(iv) Complete the table in the insert and hence find the order in which the birds should be visited to give a quickest route and find the corresponding minimum journey time. **[9]**

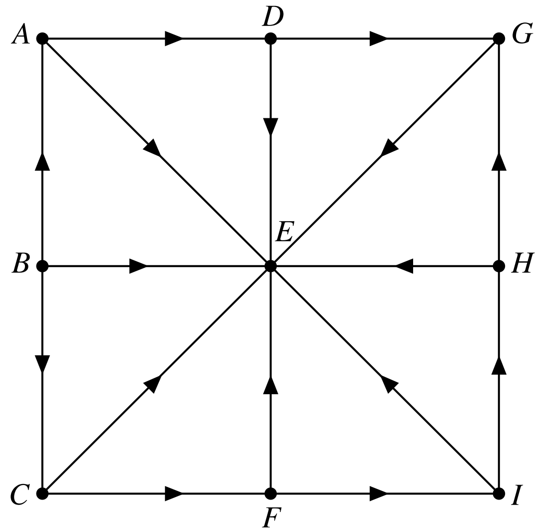
Stage	State	Action	Working	Suboptimal minimum
4	1(234)	0	10	10
	2(134)	0	14	14
	3(124)	0	12	12
	4(123)	0	17	17
3	1(23)	4(123)	17 + 6 = 23	23
	1(24)	3(124)	12 + 2 = 14	14
	1(34)	2(134)	14 + 3 = 17	17
	2(13)	4(123)	17 + 4 = 21	21
	2(14)	3(124)	12 + 2 = 14	14
	2(34)	1(234)	10 + 3 = 13	13
	3(12)	4(123)	17 + 3 = 20	20
	3(14)	2(134)	14 + 2 = 16	16
	3(24)	1(234)	10 + 2 = 12	12
	4(12)	3(124)	12 + 3 = 15	15
	4(13)	2(134)	14 + 4 = 18	18
	4(23)	1(234)	10 + 6 = 16	16
2	1(2)	3(12) 4(12)	20 + 2 = 22 15 + 6 = 21	21
	1(3)	2(13) 4(13)	21 + 3 = 24 18 + 6 = 24	24
	1(4)			
	2(1)			
	2(3)			
	2(4)			
	3(1)			
	3(2)			
	3(4)			
	4(1)			
	4(2)			
	4(3)			
1	1			
	2			
	3			
	4			
0	0	1 2 3 4		

4 (v)

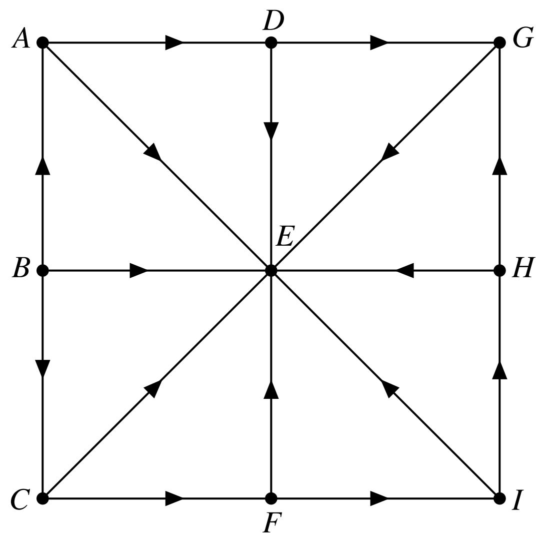
Minimum flow



Maximum flow



(vi) (a)



(b) ....

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[Answer Question 6 overleaf.]

6 **(i)** ....

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(ii) ....

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(iii) ....

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(iv) The table for this part of the question is on the opposite page.

Quickest route ....

Minimum journey time .... minutes

Stage	State	Action	Working	Suboptimal minimum
4	1(234)	0	10	10
	2(134)	0	14	14
	3(124)	0	12	12
	4(123)	0	17	17
3	1(23)	4(123)	17 + 6 = 23	23
	1(24)	3(124)	12 + 2 = 14	14
	1(34)	2(134)	14 + 3 = 17	17
	2(13)	4(123)	17 + 4 = 21	21
	2(14)	3(124)	12 + 2 = 14	14
	2(34)	1(234)	10 + 3 = 13	13
	3(12)	4(123)	17 + 3 = 20	20
	3(14)	2(134)	14 + 2 = 16	16
	3(24)	1(234)	10 + 2 = 12	12
	4(12)	3(124)	12 + 3 = 15	15
	4(13)	2(134)	14 + 4 = 18	18
	4(23)	1(234)	10 + 6 = 16	16
2	1(2)	3(12) 4(12)	20 + 2 = 22 15 + 6 = 21	21
	1(3)	2(13) 4(13)	21 + 3 = 24 18 + 6 = 24	24
	1(4)			
	2(1)			
	2(3)			
	2(4)			
	3(1)			
	3(2)			
	3(4)			
	4(1)			
	4(2)			
	4(3)			
1	1			
	2			
	3			
	4			
0	0	1 2 3 4		

1 Daniel needs to clean four houses but only has one day in which to do it. He estimates that each house will take one day and so he has asked three professional cleaning companies to give him a quotation for cleaning each house. He intends to hire the three companies to clean one house each and he will clean the fourth house himself. The table below shows the quotations that Daniel was given by the three companies.

	House 1	House 2	House 3	House 4
Allclean	£500	£400	£700	£600
Brightenupp	£300	£200	£400	£350
Clean4U	£500	£300	£750	£680

(i) Copy the table and add a dummy row to represent Daniel. [2]

(ii) Apply the Hungarian algorithm, **reducing rows first**, to find a minimum cost matching. You must show your working and say how each matrix was formed. [9]

(iii) Which house should Daniel ask each company to clean? Find the total cost of hiring the three companies. [2]

2 The table gives the pay-off matrix for a zero-sum game between two players, Amy and Bea. The values in the table show the pay-offs for Amy.

		Bea		
		Strategy X	Strategy Y	Strategy Z
Amy	Strategy P	4	−2	0
	Strategy Q	−1	5	4

Amy makes a random choice between strategies P and Q, choosing strategy P with probability p and strategy Q with probability $1 - p$.

(i) Write down and simplify an expression for the expected pay-off for Amy when Bea chooses strategy X. Write down similar expressions for the cases when Bea chooses strategy Y and when she chooses strategy Z. [4]

(ii) Using graph paper, draw a graph to show Amy's expected pay-off against p for each of Bea's choices of strategy. Using your graph, find the optimal value of p for Amy. [5]

Amy and Bea play the game many times. Amy chooses randomly between her strategies using the optimal value for p.

(iii) Showing your working, calculate Amy's minimum expected pay-off per game. Why might Amy gain more points than this, on average? [3]

(iv) What is Bea's minimum expected loss per game? How should Bea play to minimise her expected loss? [3]

3 The table shows the activities involved in a project, their durations and precedences, and the number of workers needed for each activity.

Activity	Duration (days)	Immediate predecessors	Number of workers
A	3	–	3
B	4	A	2
C	5	A	2
D	2	B, C	1
E	3	C	3
F	4	D	2
G	2	D, E	2

(i) Draw an activity network to represent the project, using activity on arc. You are advised to make the diagram quite large. The activity network requires two dummy activities; explain why each of these is needed. [4]

(ii) Carry out a forward pass to find the early times for the events. Record these at the vertices on your network. Also calculate and record the late times for the events. Find the minimum completion time for the project and list the critical activities. [6]

The number of workers required for each activity is shown in the table. Assume that each worker is able to do any of the activities. Once an activity has been started, it must run for its duration.

(iii) Using graph paper, draw a resource histogram with each activity starting at its earliest possible start time. [3]

(iv) Explain why if only four workers are available, the project cannot be completed in the minimum project completion time. Show how the project can be completed in one day more than the minimum project completion time when there are only four workers. [2]

4 Answer this question on the insert provided.

The table shows a partially completed dynamic programming tabulation for solving a minimax problem.

Stage	State	Action	Working	Minimax
1	0	0	4	4
	1	0	3	3
	2	0	2	2
2	0	0	max(6, 4) = 6	3
		1	max(2, 3) = 3	
		2	max(3, 2) = 3	
	1	0	max(2, 4) =	
		1	max(4, 3) =	
		2	max(5, 2) =	
	2	0	max(2,	
		1	max(3,	
		2	max(4,	
3	0	0	max(5,	
		1	max(5,	
		2	max(2,	

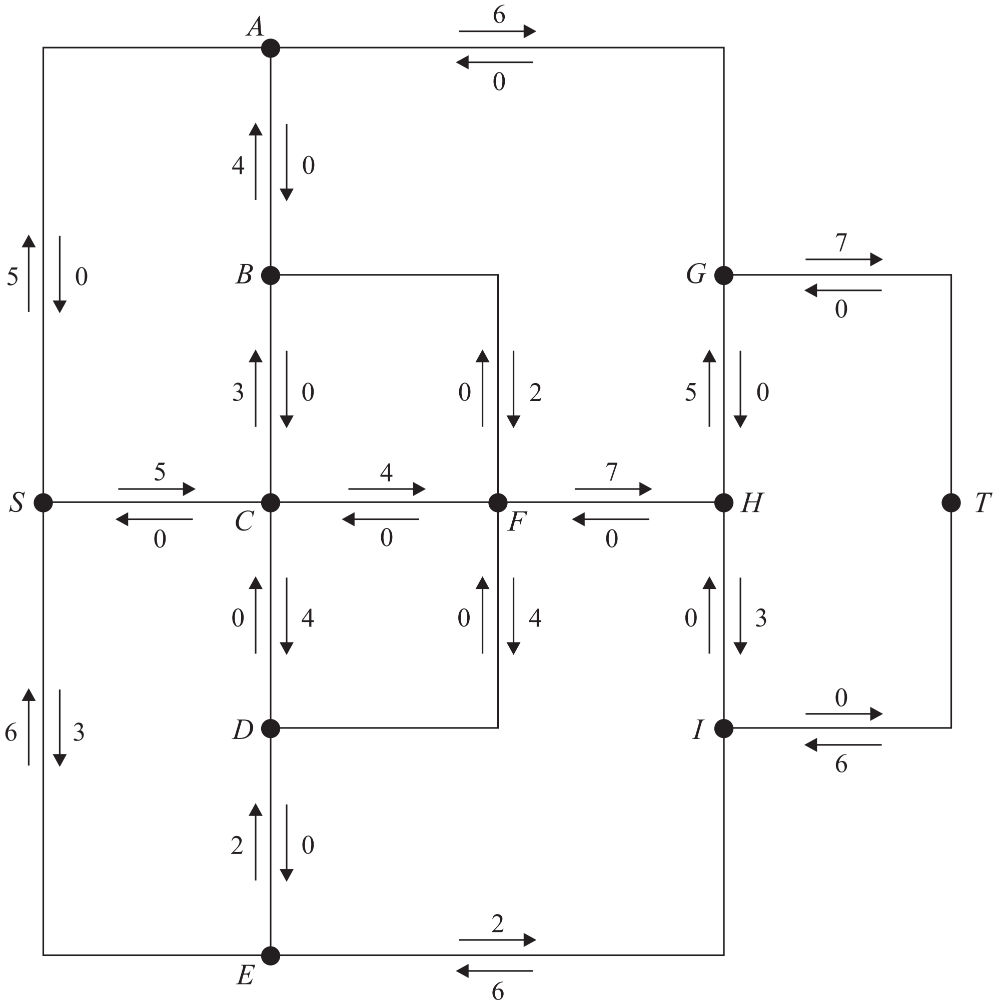
(i) On the insert, complete the last two columns of the table. [9]

(ii) State the minimax value and write down the minimax route. [4]

(iii) Complete the diagram on the insert to show the network that is represented by the table. [3]

5 Answer this question on the insert provided.

The network represents a system of pipes through which fluid can flow from a source, S, to a sink, T.



The arrows are labelled to show excess capacities and potential backflows (how much more and how much less could flow in each pipe). The excess capacities and potential backflows are measured in litres per second. Currently the flow is 6 litres per second, all flowing along a single route through the system.

(i) Write down the route of the 6 litres per second that is flowing from S to T. [1]

(ii) What is the capacity of the pipe AG and in which direction can fluid flow along this pipe? [2]

(iii) Calculate the capacity of the cut X = $\{S, A, B, C, D, E\}$, Y = $\{F, G, H, I, T\}$. [3]

(iv) Describe how a further 7 litres per second can flow from S to T and update the labels on the arrows to show your flow. Explain how you know that this is the maximum flow. [7]

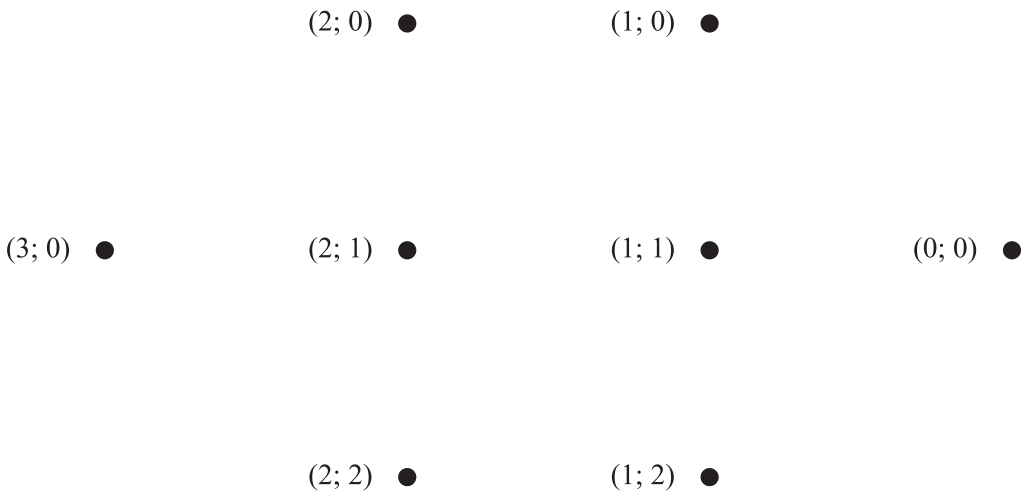
4 **(i)**

Stage	State	Action	Working	Minimax
1	0	0	4	4
	1	0	3	3
	2	0	2	2
2	0	0	max(6, 4) = 6	3
		1	max(2, 3) = 3	
		2	max(3, 2) = 3	
	1	0	max(2, 4) =	
		1	max(4, 3) =	
		2	max(5, 2) =	
	2	0	max(2,	
		1	max(3,	
		2	max(4,	
3	0	0	max(5,	
		1	max(5,	
		2	max(2,	

(ii) Minimax value = ...

Route = ...

(iii)



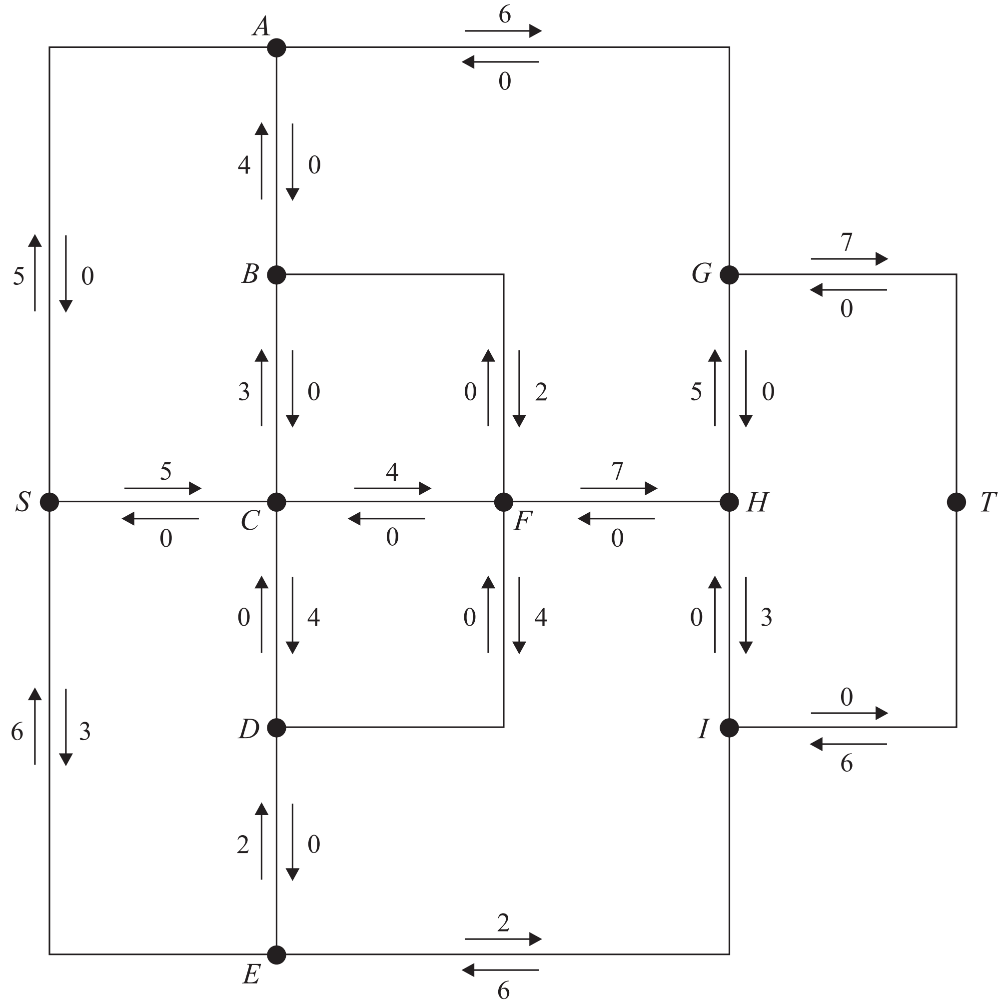
5 **(i)** ....

(ii) ....

(iii) ....

(iv) ....

....



1 Answer this question on the insert provided.

The table shows a partially completed dynamic programming tabulation for solving a maximin problem.

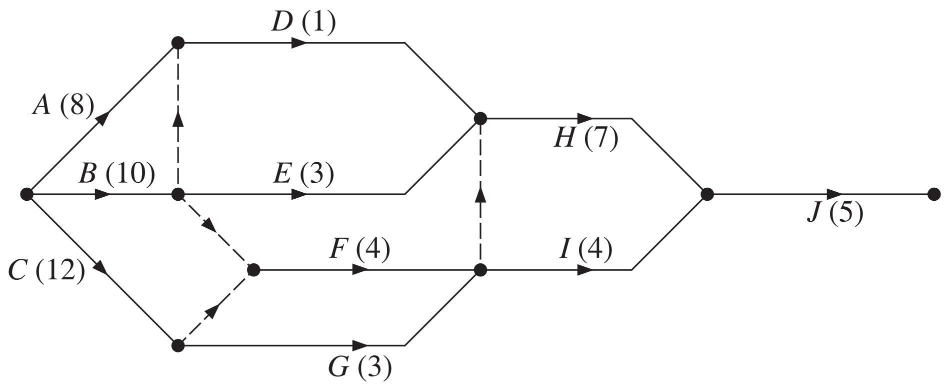
Stage	State	Action	Working	Maximin
1	0	0	10	
	1	0	11	
	2	0	14	
	3	0	15	
2	0	0	(12,) =	
		2	(10,) =	
	1	0	(13,) =	
		1	(10,) =	
		2	(11,) =	
	2	1	(9,) =	
		2	(10,) =	
		3	(7,) =	
	3	1	(8,) =	
		3	(12,) =	
3	0	0	(15,) =	
		1	(14,) =	
		2	(16,) =	
		3	(13,) =	

(i) Complete the last two columns of the table in the insert. **[6]**

(ii) State the maximin value and write down the maximin route. **[3]**

2 **Answer this question on the insert provided.**

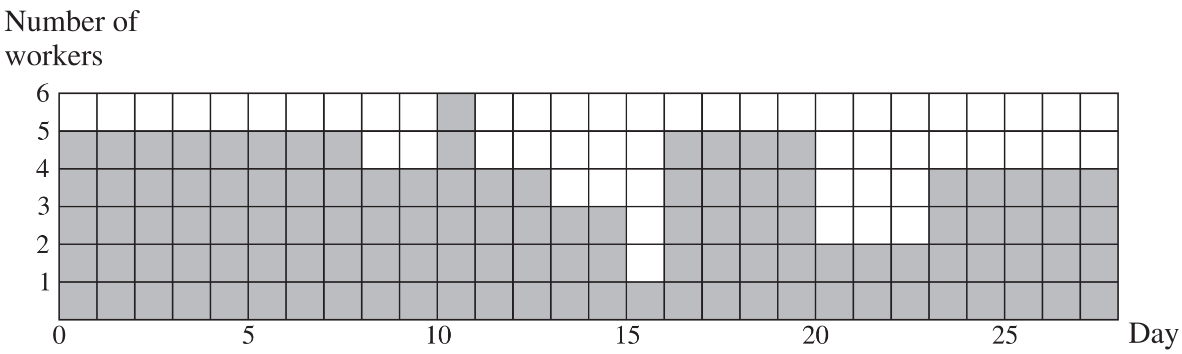
The diagram shows an activity network for a project. The figures in brackets show the durations of the activities in days.



(i) Complete the table in the insert to show the precedences for the activities. **[3]**

(ii) Use the boxes on the diagram in the insert to carry out a forward pass and a backward pass. Show that the minimum project completion time is 28 days and list the critical activities. **[4]**

The resource histogram below shows the number of workers required each day when the activities each begin at their earliest possible start time. Once an activity has been started it runs for its duration without a break.



(iii) By considering which activities are happening each day, complete the table in the insert to show the number of workers required for each activity. You are advised to start at day 28 and work back through the days towards day 1. **[6]**

Only five workers are actually available, but they are all equally skilled at each of the activities. The project can still be completed in 28 days by delaying the start of activity E.

(iv) Find the minimum possible delay and the maximum possible delay on activity E in this case. **[2]**

3 **Answer this question on the insert provided.**

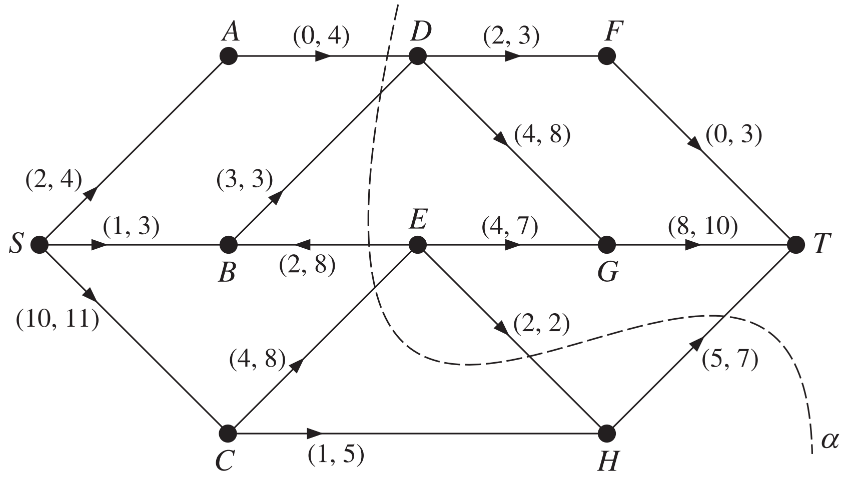


Fig. 1

Fig. 1 represents a system of pipes through which fluid can flow from a source, S, to a sink, T. It also shows a cut α. The weights on the arcs show the lower and upper capacities of the pipes in litres per second.

(i) Calculate the capacity of the cut α. **[2]**

(ii) By considering vertex B, explain why arc SB must be at its lower capacity. Then by considering vertex E, explain why arc CE must be at its upper capacity, and hence explain why arc HT must be at its lower capacity. **[4]**

(iii) On the diagram in the insert, show a flow through the network of 15 litres per second. Write down one flow augmenting route that allows another 1 litre per second to flow through the network. Show that the maximum flow is 16 litres per second by finding a cut of 16 litres per second. **[4]**

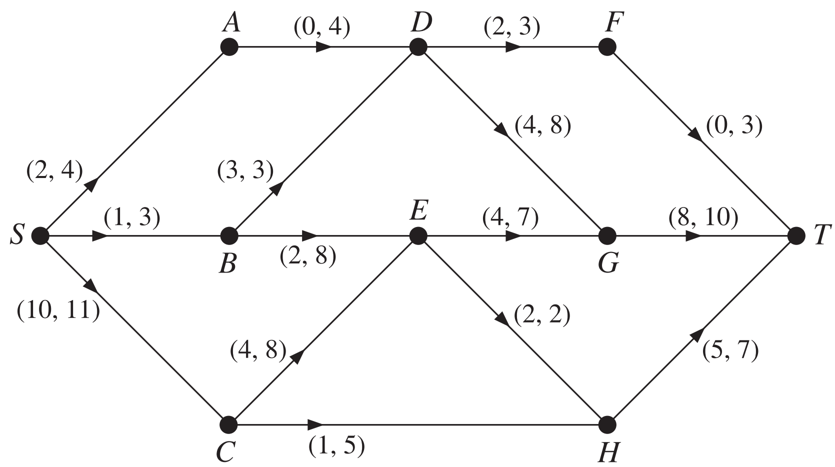


Fig. 2

Fig. 2 represents the same system, but with pipe EB installed the wrong way round.

(iv) Explain why there can be no feasible flow through this network. **[2]**

4 Anya (A), Ben (B), Connie (C), Derek (D) and Emma (E) work for a local newspaper. The editor wants them each to write a regular weekly article for the paper. The items needed are: problem page (P), restaurant review (R), sports news (S), theatre review (T) and weather report (W).

Anya wants to write either the problem page or the restaurant review. She is given the problem page.

Ben wants the restaurant review, the sports news or the theatre review. The editor gives him the restaurant review.

Connie wants either the theatre review or the weather report. The editor gives her the theatre review.

Derek wants the problem page, the sports news or the weather report. He is given the weather report.

Emma is only interested in writing the problem page but this has already been given to Anya.

(i) Draw a bipartite graph to show the possible pairings between the writers (A, B, C, D and E) and the articles (P, R, S, T and W). On your bipartite graph, show who has been given which article by the editor. **[2]**

(ii) Construct the shortest possible alternating path, starting from Emma, to find a complete matching between the writers and the articles. Write a list showing which article each writer is given with this complete matching. **[3]**

When the writers send in their articles the editor assigns a sub-editor to each one to check it. The sub-editors can check at most one article each.

The table shows how long, in minutes, each sub-editor would typically take to check each article.

		Article				
		P	R	S	T	W
Sub-editor	Jeremy (J)	56	56	51	57	58
	Kath (K)	53	52	53	54	54
	Laura (L)	57	55	52	58	60
	Mohammed (M)	59	55	53	59	57
	Natalie (N)	57	57	53	59	60
	Ollie (O)	58	56	51	56	57

The editor wants to find the allocation for which the total time spent checking the articles is as short as possible.

(iii) Apply the Hungarian algorithm to the table, reducing rows first, to find an optimal allocation between the sub-editors and the articles. Explain how each table is formed and write a list showing which sub-editor should be assigned to which article. If each minute of sub-editor time costs £0.25, calculate the total cost of checking the articles each week. **[11]**

[Question 5 is printed overleaf.]

5 The local rugby club has challenged the local cricket club to a chess match to raise money for charity. Each of the top three chess players from the rugby club has played 10 chess games against each of the top three chess players from the cricket club. There were no drawn games. The table shows, for each pairing, the number of games won by the player from the rugby club minus the number of games won by the player from the cricket club. This will be called the score; the scores make a zero-sum game.

		Cricket club		
		Doug	Euan	Fiona
Rugby club	Sanjeev	0	4	−2
	Tom	−4	2	−4
	Ursula	2	−6	0

(i) How many of the 10 games between Sanjeev and Doug did Sanjeev win? How many of the 10 games between Sanjeev and Euan did Euan win? **[3]**

Each club must choose one person to play. They want to choose the person who will optimise the score.

(ii) Find the play-safe choice for each club, showing your working. Explain how you know that the game is not stable. **[5]**

(iii) Which person should the cricket club choose if they know that the rugby club will play-safe and which person should the rugby club choose if they know that the cricket club will play-safe? **[2]**

(iv) Explain why the rugby club should not choose Tom. Which player should the cricket club not choose, and why? **[3]**

The rugby club chooses its player by using random numbers to choose between Sanjeev and Ursula, where the probability of choosing Sanjeev is p and the probability of choosing Ursula is $1 - p$.

(v) Write down an expression for the expected score for the rugby club for each of the two remaining choices that can be made by the cricket club. Calculate the optimal value for p. **[2]**

Doug is studying AS Mathematics. He removes the row representing Tom and then models the cricket club's problem as the following LP.

$$\begin{array}{ll} \text{maximise} & M = m - 4 \\ \text{subject to} & m \leqslant 4x \phantom{{}+10y} + 6z \\ & m \leqslant 2x + 10y + 4z \\ & x + y + z \leqslant 1 \\ \text{and} & m \geqslant 0,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0 \end{array}$$

(vi) Show how Doug used the values in the table to get the constraints $m \leqslant 4x + 6z$ and $m \leqslant 2x + 10y + 4z$. **[3]**

Doug uses the Simplex algorithm to solve the LP problem. His solution has $x = 0$ and $y = \frac{1}{6}$.

(vii) Calculate the optimal value of M. **[2]**

1 **(i)**

Stage	State	Action	Working	Maximin
1	0	0	10	
	1	0	11	
	2	0	14	
	3	0	15	
2	0	0	(12,) =	
		2	(10,) =	
	1	0	(13,) =	
		1	(10,) =	
		2	(11,) =	
	2	1	(9,) =	
		2	(10,) =	
		3	(7,) =	
	3	1	(8,) =	
		3	(12,) =	
3	0	0	(15,) =	
		1	(14,) =	
		2	(16,) =	
		3	(13,) =	

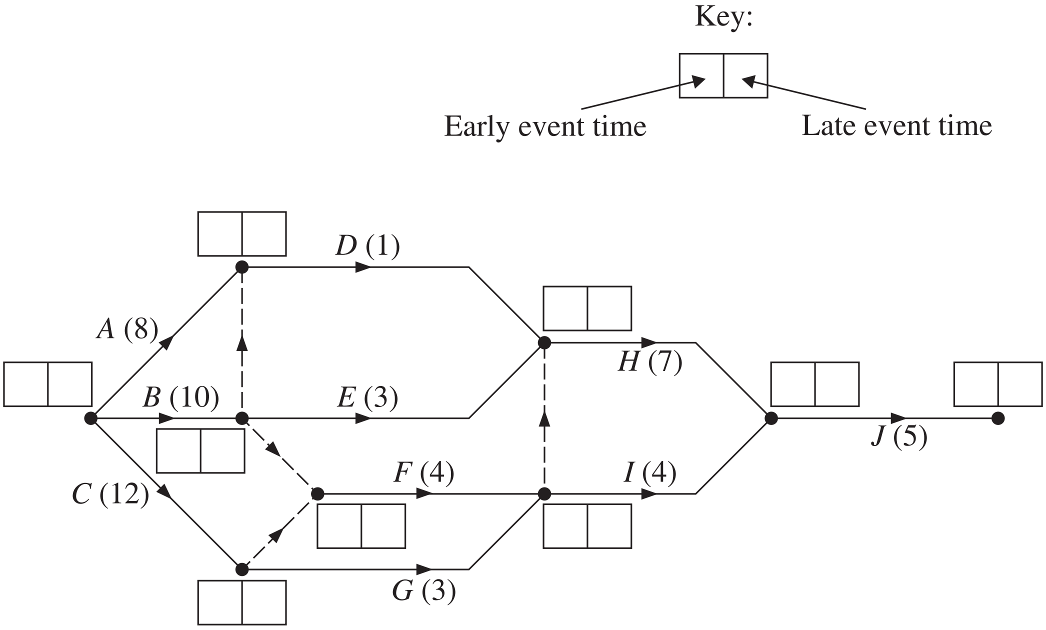
(ii) Maximin value = ...

Maximin route = ...

2 **(i)**

Activity	Duration (days)	Immediate predecessors
A	8	
B	10	
C	12	
D	1	
E	3	
F	4	
G	3	
H	7	
I	4	
J	5	

(ii)



Critical activities ....

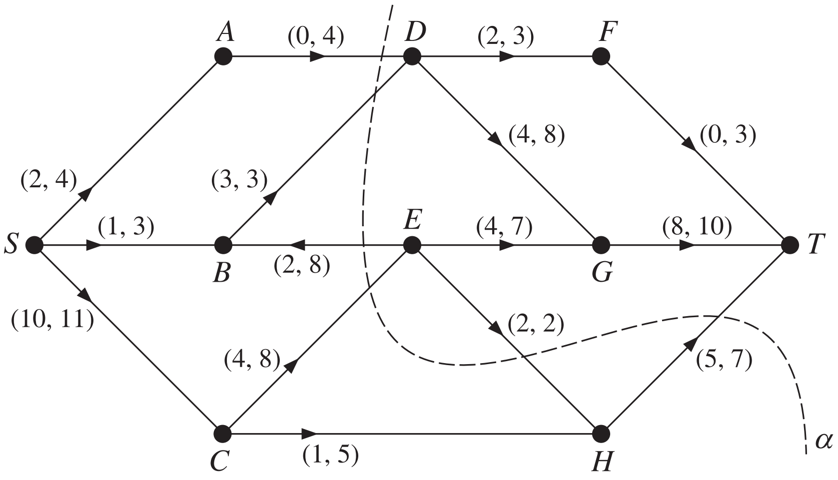
(iii)

Activity	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*	*J*
Number of workers										

(iv) Minimum delay .... days

Maximum delay .... days

3



(i) Capacity of cut α = ... litres per second

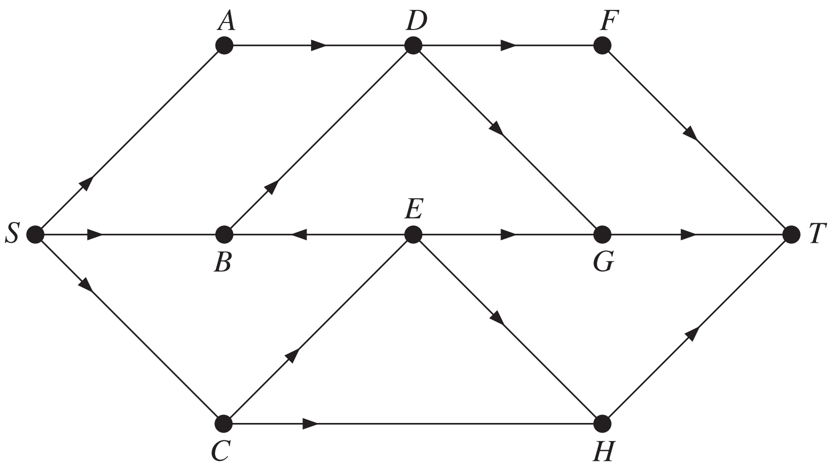
(ii) ....

....

....

....

(iii)



Flow augmenting route: ....

Cut: ....

(iv) ....

....