Answer **all** questions.

1 (a) Find the roots of the equation $m^2 + 2m + 2 = 0$ in the form $a + \mathrm{i}b$. *(2 marks)*

(b) (i) Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 4x$$

(6 marks)

(ii) Hence express y in terms of x, given that $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$. *(4 marks)*

2 (a) Find $\int_0^a x\mathrm{e}^{-2x}\,\mathrm{d}x$, where $a > 0$. *(5 marks)*

(b) Write down the value of $\lim_{a\to\infty} a^k\mathrm{e}^{-2a}$, where k is a positive constant. *(1 mark)*

(c) Hence find $\int_0^\infty x\mathrm{e}^{-2x}\,\mathrm{d}x$. *(2 marks)*

3 (a) Show that $y = x^3 - x$ is a particular integral of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2xy}{x^2 - 1} = 5x^2 - 1$$

(3 marks)

(b) By differentiating $(x^2 - 1)y = c$ implicitly, where y is a function of x and c is a constant, show that $y = \frac{c}{x^2 - 1}$ is a solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2xy}{x^2 - 1} = 0$$

(3 marks)

(c) Hence find the general solution of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2xy}{x^2 - 1} = 5x^2 - 1$$

(2 marks)

4 (a) Use the series expansion

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$$

to write down the first four terms in the expansion, in ascending powers of x, of $\ln(1-x)$. *(1 mark)*

(b) The function f is defined by

$$f(x) = e^{\sin x}$$

Use Maclaurin's theorem to show that when $f(x)$ is expanded in ascending powers of x:

(i) the first three terms are

$$1 + x + \frac{1}{2}x^2$$ *(6 marks)*

(ii) the coefficient of x^3 is zero. *(3 marks)*

(c) Find

$$\lim_{x \to 0} \frac{e^{\sin x} - 1 + \ln(1-x)}{x^2 \sin x}$$ *(4 marks)*

Turn over for the next question

5 (a) The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x,y) = x\ln x + \frac{y}{x}$$

and $$y(1) = 1$$

(i) Use the Euler formula

$$y_{r+1} = y_r + h\,f(x_r\,, y_r)$$

with $h = 0.1$, to obtain an approximation to $y(1.1)$. *(3 marks)*

(ii) Use the formula

$$y_{r+1} = y_{r-1} + 2h\,f(x_r\,, y_r)$$

with your answer to part (a)(i) to obtain an approximation to $y(1.2)$, giving your answer to three decimal places. *(4 marks)*

(b) (i) Show that $\frac{1}{x}$ is an integrating factor for the first-order differential equation

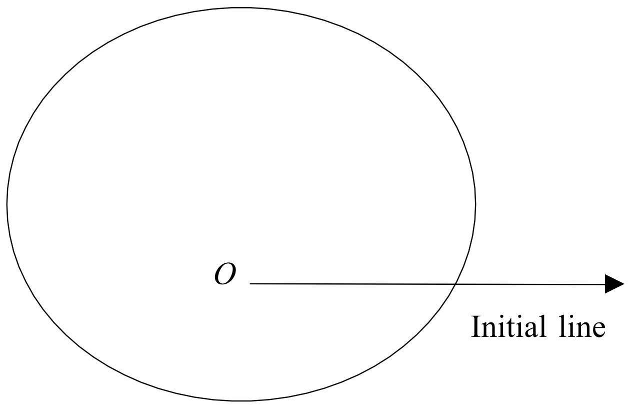
$$\frac{dy}{dx} - \frac{1}{x}y = x\ln x$$ *(3 marks)*

(ii) Solve this differential equation, given that $y = 1$ when $x = 1$. *(6 marks)*

(iii) Calculate the value of y when $x = 1.2$, giving your answer to three decimal places. *(1 mark)*

6 (a) A circle C_1 has cartesian equation $x^2 + (y - 6)^2 = 36$. Show that the polar equation of C_1 is $r = 12\sin\theta$. *(4 marks)*

(b) A curve C_2 with polar equation $r = 2\sin\theta + 5,\ 0 \leqslant \theta \leqslant 2\pi$ is shown in the diagram.



Calculate the area bounded by C_2. *(6 marks)*

(c) The circle C_1 intersects the curve C_2 at the points P and Q. Find, in surd form, the area of the quadrilateral $OPMQ$, where M is the centre of the circle and O is the pole. *(6 marks)*

END OF QUESTIONS

Answer **all** questions.

1 It is given that y satisfies the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 8x - 10 - 10\cos 2x$$

(a) Show that $y = 2x + \sin 2x$ is a particular integral of the given differential equation. *(3 marks)*

(b) Find the general solution of the differential equation. *(4 marks)*

(c) Hence express y in terms of x, given that $y = 2$ and $\frac{dy}{dx} = 0$ when $x = 0$. *(4 marks)*

2 The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where
$$f(x, y) = \frac{x^2 + y^2}{xy}$$

and
$$y(1) = 2$$

(a) Use the Euler formula

$$y_{r+1} = y_r + hf(x_r, y_r)$$

with $h = 0.1$, to obtain an approximation to $y(1.1)$. *(3 marks)*

(b) Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_r, y_r)$ and $k_2 = hf(x_r + h, y_r + k_1)$ and $h = 0.1$, to obtain an approximation to $y(1.1)$, giving your answer to four decimal places. *(6 marks)*

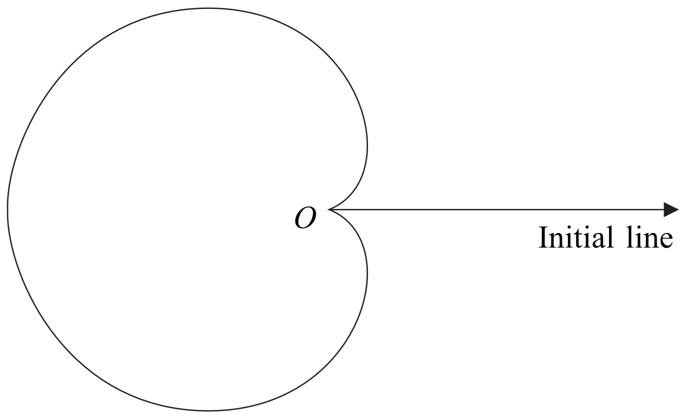
3 (a) Show that $\sin x$ is an integrating factor for the differential equation

$$\frac{dy}{dx} + (\cot x)y = 2\cos x$$ *(3 marks)*

(b) Solve this differential equation, given that $y = 2$ when $x = \frac{\pi}{2}$. *(6 marks)*

4 The diagram shows the curve C with polar equation

$$r = 6(1 - \cos\theta), \qquad 0 \leqslant \theta < 2\pi$$



(a) Find the area of the region bounded by the curve C. *(6 marks)*

(b) The circle with cartesian equation $x^2 + y^2 = 9$ intersects the curve C at the points A and B.

(i) Find the polar coordinates of A and B. *(4 marks)*

(ii) Find, in surd form, the length of AB. *(2 marks)*

5 (a) Show that $\lim\limits_{a\to\infty}\left(\frac{3a+2}{2a+3}\right) = \frac{3}{2}$. *(2 marks)*

(b) Evaluate $\int_1^{\infty}\left(\frac{3}{3x+2} - \frac{2}{2x+3}\right)dx$, giving your answer in the form $\ln k$, where k is a rational number. *(5 marks)*

6 (a) Show that the substitution

$$u = \frac{dy}{dx} + 2y$$

transforms the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$$

into

$$\frac{du}{dx} + 2u = e^{-2x}$$ *(4 marks)*

(b) By using an integrating factor, or otherwise, find the general solution of

$$\frac{du}{dx} + 2u = e^{-2x}$$

giving your answer in the form $u = f(x)$. *(5 marks)*

(c) Hence find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$$

giving your answer in the form $y = g(x)$. *(5 marks)*

7 (a) (i) Write down the first three terms of the binomial expansion of $(1+y)^{-1}$, in ascending powers of y. *(1 mark)*

(ii) By using the expansion

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

and your answer to part (a)(i), or otherwise, show that the first three non-zero terms in the expansion, in ascending powers of x, of $\sec x$ are

$$1 + \frac{x^2}{2} + \frac{5x^4}{24}$$ *(5 marks)*

(b) By using Maclaurin's theorem, or otherwise, show that the first two non-zero terms in the expansion, in ascending powers of x, of $\tan x$ are

$$x + \frac{x^3}{3}$$ *(3 marks)*

(c) Hence find $\lim_{x \to 0} \left(\dfrac{x \tan 2x}{\sec x - 1} \right)$. *(4 marks)*

END OF QUESTIONS

Answer **all** questions.

1 The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x, y) = \ln(1 + x^2 + y)$$

and $$y(1) = 0.6$$

(a) Use the Euler formula

$$y_{r+1} = y_r + h\,f(x_r, y_r)$$

with $h = 0.05$, to obtain an approximation to $y(1.05)$, giving your answer to four decimal places. *(3 marks)*

(b) Use the improved Euler formula

$$y_{r+1} = y_r + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = h\,f(x_r, y_r)$ and $k_2 = h\,f(x_r + h, y_r + k_1)$ and $h = 0.05$, to obtain an approximation to $y(1.05)$, giving your answer to four decimal places. *(6 marks)*

2 A curve has polar equation $r(1 - \sin\theta) = 4$. Find its cartesian equation in the form $y = f(x)$. *(6 marks)*

3 (a) Show that x^2 is an integrating factor for the first-order differential equation

$$\frac{dy}{dx} + \frac{2}{x}y = 3(x^3 + 1)^{\frac{1}{2}}$$ *(3 marks)*

(b) Solve this differential equation, given that $y = 1$ when $x = 2$. *(6 marks)*

4 (a) Explain why $\int_0^{e} \frac{\ln x}{\sqrt{x}}\, dx$ is an improper integral. *(1 mark)*

(b) Use integration by parts to find $\int x^{-\frac{1}{2}} \ln x \, dx$. *(3 marks)*

(c) Show that $\int_0^{e} \frac{\ln x}{\sqrt{x}}\, dx$ exists and find its value. *(4 marks)*

5 Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 6 + 5\sin x$$ *(12 marks)*

6 The function f is defined by $f(x) = (1 + 2x)^{\frac{1}{2}}$.

(a) (i) Find $f'''(x)$. *(4 marks)*

(ii) Using Maclaurin's theorem, show that, for small values of x,

$$f(x) \approx 1 + x - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3$$ *(4 marks)*

(b) Use the expansion of e^x together with the result in part (a)(ii) to show that, for small values of x,

$$e^x(1 + 2x)^{\frac{1}{2}} \approx 1 + 2x + x^2 + kx^3$$

where k is a rational number to be found. *(3 marks)*

(c) Write down the first four terms in the expansion, in ascending powers of x, of e^{2x}. *(1 mark)*

(d) Find

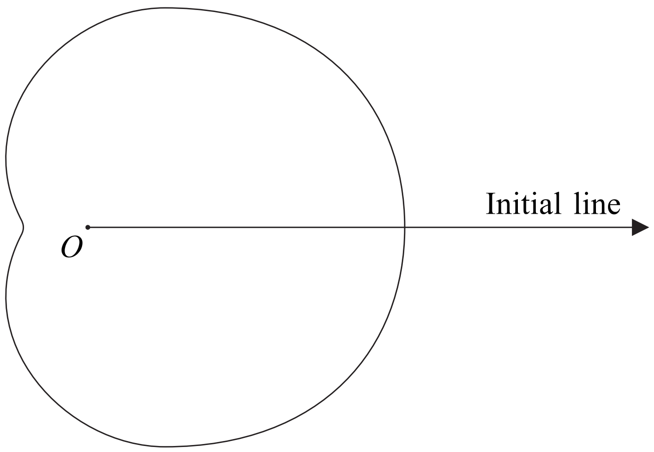
$$\lim_{x \to 0} \frac{e^x(1 + 2x)^{\frac{1}{2}} - e^{2x}}{1 - \cos x}$$ *(4 marks)*

Turn over for the next question

7 A curve C has polar equation

$$r = 6 + 4\cos\theta, \qquad -\pi \leqslant \theta \leqslant \pi$$

The diagram shows a sketch of the curve C, the pole O and the initial line.



(a) Calculate the area of the region bounded by the curve C. *(6 marks)*

(b) The point P is the point on the curve C for which $\theta = \frac{2\pi}{3}$.

The point Q is the point on C for which $\theta = \pi$.

Show that QP is parallel to the line $\theta = \frac{\pi}{2}$. *(4 marks)*

(c) The line PQ intersects the curve C again at a point R.

The line RO intersects C again at a point S.

(i) Find, in surd form, the length of PS. *(4 marks)*

(ii) Show that the angle OPS is a right angle. *(1 mark)*

END OF QUESTIONS

Answer **all** questions.

1 (a) Find the value of the constant k for which kx^2e^{5x} is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 6e^{5x}$$ *(6 marks)*

(b) Hence find the general solution of this differential equation. *(4 marks)*

2 The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x, y) = \sqrt{x^2 + y^2 + 3}$$

and $$y(1) = 2$$

(a) Use the Euler formula

$$y_{r+1} = y_r + hf(x_r, y_r)$$

with $h = 0.1$, to obtain an approximation to $y(1.1)$, giving your answer to four decimal places. *(3 marks)*

(b) Use the improved Euler formula

$$y_{r+1} = y_r + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_r, y_r)$ and $k_2 = hf(x_r + h, y_r + k_1)$ and $h = 0.1$, to obtain an approximation to $y(1.1)$, giving your answer to four decimal places. *(6 marks)*

3 By using an integrating factor, find the solution of the differential equation

$$\frac{dy}{dx} + (\tan x)y = \sec x$$

given that $y = 3$ when $x = 0$. *(8 marks)*

4 (a) Show that $(\cos\theta + \sin\theta)^2 = 1 + \sin 2\theta$. *(1 mark)*

(b) A curve has cartesian equation

$$(x^2 + y^2)^3 = (x + y)^4$$

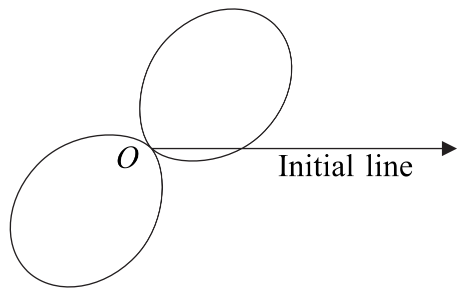
Given that $r \geqslant 0$, show that the polar equation of the curve is

$$r = 1 + \sin 2\theta$$ *(4 marks)*

(c) The curve with polar equation

$$r = 1 + \sin 2\theta, \quad -\pi \leqslant \theta \leqslant \pi$$

is shown in the diagram.



(i) Find the two values of θ for which $r = 0$. *(3 marks)*

(ii) Find the area of one of the loops. *(6 marks)*

Turn over for the next question

5 (a) A differential equation is given by

$$(x^2 - 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = x^2 + 1$$

Show that the substitution

$$u = \frac{dy}{dx} + x$$

transforms this differential equation into

$$\frac{du}{dx} = \frac{2xu}{x^2 - 1}$$

(4 marks)

(b) Find the general solution of

$$\frac{du}{dx} = \frac{2xu}{x^2 - 1}$$

giving your answer in the form $u = f(x)$. *(5 marks)*

(c) Hence find the general solution of the differential equation

$$(x^2 - 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = x^2 + 1$$

giving your answer in the form $y = g(x)$. *(3 marks)*

6 (a) The function f is defined by

$$\mathrm{f}(x) = \ln(1 + \mathrm{e}^x)$$

Use Maclaurin's theorem to show that when $\mathrm{f}(x)$ is expanded in ascending powers of x:

(i) the first three terms are

$$\ln 2 + \tfrac{1}{2}x + \tfrac{1}{8}x^2$$ *(6 marks)*

(ii) the coefficient of x^3 is zero. *(3 marks)*

(b) Hence write down the first two non-zero terms in the expansion, in ascending powers of x, of $\ln\left(\dfrac{1 + \mathrm{e}^x}{2}\right)$. *(1 mark)*

(c) Use the series expansion

$$\ln(1 + x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \ldots$$

to write down the first three terms in the expansion, in ascending powers of x, of $\ln\left(1 - \dfrac{x}{2}\right)$. *(1 mark)*

(d) Use your answers to parts (b) and (c) to find

$$\lim_{x \to 0} \left[\frac{\ln\left(\dfrac{1 + \mathrm{e}^x}{2}\right) + \ln\left(1 - \dfrac{x}{2}\right)}{x - \sin x} \right]$$ *(4 marks)*

7 (a) Write down the value of

$$\lim_{x \to \infty} x\mathrm{e}^{-x}$$ *(1 mark)*

(b) Use the substitution $u = x\mathrm{e}^{-x} + 1$ to find $\displaystyle\int \frac{\mathrm{e}^{-x}(1 - x)}{x\mathrm{e}^{-x} + 1}\,\mathrm{d}x$. *(2 marks)*

(c) Hence evaluate $\displaystyle\int_1^{\infty} \frac{1 - x}{x + \mathrm{e}^x}\,\mathrm{d}x$, showing the limiting process used. *(4 marks)*

END OF QUESTIONS

Answer **all** questions.

1 The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x, y) = x^2 - y^2$$

and $$y(2) = 1$$

(a) Use the Euler formula

$$y_{r+1} = y_r + hf(x_r, y_r)$$

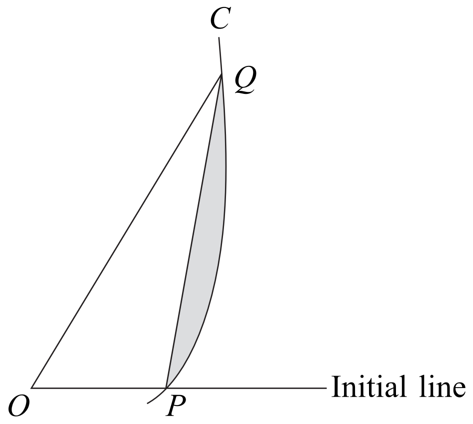
with $h = 0.1$, to obtain an approximation to $y(2.1)$. *(3 marks)*

(b) Use the formula

$$y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$$

with your answer to part (a), to obtain an approximation to $y(2.2)$. *(3 marks)*

2 The diagram shows a sketch of part of the curve C whose polar equation is $r = 1 + \tan\theta$. The point O is the pole.



The points P and Q on the curve are given by $\theta = 0$ and $\theta = \frac{\pi}{3}$ respectively.

(a) Show that the area of the region bounded by the curve C and the lines OP and OQ is

$$\frac{1}{2}\sqrt{3} + \ln 2$$ *(6 marks)*

(b) Hence find the area of the shaded region bounded by the line PQ and the arc PQ of C. *(3 marks)*

3 (a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 5$$ *(6 marks)*

(b) Hence express y in terms of x, given that $y = 2$ and $\frac{dy}{dx} = 3$ when $x = 0$. *(4 marks)*

4 (a) Explain why $\int_1^{\infty} xe^{-3x}\,dx$ is an improper integral. *(1 mark)*

(b) Find $\int xe^{-3x}\,dx$. *(3 marks)*

(c) Hence evaluate $\int_1^{\infty} xe^{-3x}\,dx$, showing the limiting process used. *(3 marks)*

5 By using an integrating factor, find the solution of the differential equation

$$\frac{dy}{dx} + \frac{4x}{x^2+1}y = x$$

given that $y = 1$ when $x = 0$. Give your answer in the form $y = f(x)$. *(9 marks)*

6 A curve C has polar equation

$$r^2 \sin 2\theta = 8$$

(a) Find the cartesian equation of C in the form $y = f(x)$. *(3 marks)*

(b) Sketch the curve C. *(1 mark)*

(c) The line with polar equation $r = 2\sec\theta$ intersects C at the point A. Find the polar coordinates of A. *(4 marks)*

7 (a) (i) Write down the expansion of $\ln(1+2x)$ in ascending powers of x up to and including the term in x^3. *(2 marks)*

(ii) State the range of values of x for which this expansion is valid. *(1 mark)*

(b) (i) Given that $y = \ln\cos x$, find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$. *(4 marks)*

(ii) Find the value of $\frac{d^4y}{dx^4}$ when $x = 0$. *(3 marks)*

(iii) Hence, by using Maclaurin's theorem, show that the first two non-zero terms in the expansion, in ascending powers of x, of $\ln\cos x$ are

$$-\frac{x^2}{2} - \frac{x^4}{12}$$ *(2 marks)*

(c) Find

$$\lim_{x \to 0}\left[\frac{x\ln(1+2x)}{x^2 - \ln\cos x}\right]$$ *(3 marks)*

8 (a) Given that $x = \mathrm{e}^t$ and that y is a function of x, show that:

(i) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}t}$; *(3 marks)*

(ii) $x^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \dfrac{\mathrm{d}^2y}{\mathrm{d}t^2} - \dfrac{\mathrm{d}y}{\mathrm{d}t}$. *(3 marks)*

(b) Hence find the general solution of the differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6x\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0$$ *(5 marks)*

END OF QUESTIONS