Answer **all** questions.

1 Describe the geometrical transformation defined by the matrix

$$\begin{bmatrix} 0.6 & 0 & 0.8 \\ 0 & 1 & 0 \\ -0.8 & 0 & 0.6 \end{bmatrix}$$ *(3 marks)*

2 The matrices **P** and **Q** are defined in terms of the constant k by

$$\mathbf{P} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & k \\ 5 & 3 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} 5 & 4 & 1 \\ 3 & k & -1 \\ 7 & 3 & 2 \end{bmatrix}$$

(a) Express $\det \mathbf{P}$ and $\det \mathbf{Q}$ in terms of k. *(3 marks)*

(b) Given that $\det(\mathbf{PQ}) = 16$, find the two possible values of k. *(4 marks)*

3 (a) The plane Π has equation $\mathbf{r} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$.

(i) Find a vector which is perpendicular to both $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$. *(2 marks)*

(ii) Hence find an equation for Π in the form $\mathbf{r} \cdot \mathbf{n} = d$. *(2 marks)*

(b) The line L has equation $\left(\mathbf{r} - \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}\right) \times \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \mathbf{0}$.

Verify that $\mathbf{r} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ is also an equation for L. *(2 marks)*

(c) Determine the position vector of the point of intersection of Π and L. *(4 marks)*

4 The vectors **a**, **b** and **c** are given by

$$\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} \quad \text{and} \quad \mathbf{c} = 4\mathbf{i} - \mathbf{j} + 5\mathbf{k}$$

(a) (i) Evaluate $\begin{vmatrix} 1 & -1 & -1 \\ 2 & 3 & -1 \\ 4 & -1 & 5 \end{vmatrix}$. *(2 marks)*

(ii) Hence determine whether **a**, **b** and **c** are linearly dependent or independent. *(1 mark)*

(b) (i) Evaluate $\mathbf{b} \cdot \mathbf{c}$. *(2 marks)*

(ii) Show that $\mathbf{b} \times \mathbf{c}$ can be expressed in the form $m\mathbf{a}$, where m is a scalar. *(2 marks)*

(iii) Use these results to describe the geometrical relationship between **a**, **b** and **c**. *(1 mark)*

(c) The points A, B and C have position vectors **a**, **b** and **c** respectively relative to an origin O. The points O, A, B and C are four of the eight vertices of a cuboid. Determine the volume of this cuboid. *(2 marks)*

5 The transformation T maps (x, y) to (x', y'), where

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(a) Describe the difference between *an invariant line* and *a line of invariant points* of T. *(1 mark)*

(b) Evaluate the determinant of the matrix $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and describe the geometrical significance of the result in relation to T. *(2 marks)*

(c) Show that T has a line of invariant points, and find a cartesian equation for this line. *(2 marks)*

(d) (i) Find the image of the point $(x, -x + c)$ under T. *(2 marks)*

(ii) Hence show that all lines of the form $y = -x + c$, where c is an arbitrary constant, are invariant lines of T. *(2 marks)*

(e) Describe the transformation T geometrically. *(3 marks)*

6 (a) Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix} = (a-b)(b-c)(c-a)$$ *(5 marks)*

(b) (i) Hence, or otherwise, show that the system of equations

$$\begin{aligned} x + \quad y + \quad z &= p \\ 3x + \ 3y + 5z &= q \\ 15x + 15y + 9z &= r \end{aligned}$$

has no unique solution whatever the values of p, q and r. *(2 marks)*

(ii) Verify that this system is consistent when $24p - 3q - r = 0$. *(2 marks)*

(iii) Find the solution of the system in the case where $p = 1$, $q = 8$ and $r = 0$. *(5 marks)*

7 The matrix $\mathbf{M} = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -3 & 1 \\ 3 & -5 & 3 \end{bmatrix}$.

(a) Given that $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are eigenvectors of **M**, find the eigenvalues corresponding to **u** and **v**. *(5 marks)*

(b) Given also that the third eigenvalue of **M** is 1, find a corresponding eigenvector. *(6 marks)*

(c) (i) Express the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in terms of **u** and **v**. *(1 mark)*

(ii) Deduce that $\mathbf{M}^n \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \lambda^n \mathbf{u} + \mu^n \mathbf{v}$, where λ and μ are scalar constants whose values should be stated. *(4 marks)*

(iii) Hence prove that, for all positive **odd** integers n,

$$\mathbf{M}^n \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2^n \\ 0 \\ 2^n \end{bmatrix}$$ *(3 marks)*

END OF QUESTIONS

Answer **all** questions.

1 Two planes, Π_1 and Π_2, have equations $\mathbf{r} \cdot \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} = 0$ and $\mathbf{r} \cdot \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = 0$ respectively.

(a) Determine the cosine of the acute angle between Π_1 and Π_2. *(4 marks)*

(b) (i) Find $\begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$. *(2 marks)*

(ii) Find a vector equation for the line of intersection of Π_1 and Π_2. *(2 marks)*

2 A transformation is represented by the matrix $\mathbf{A} = \begin{bmatrix} 0.28 & -0.96 & 0 \\ 0.96 & 0.28 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Evaluate det $\mathbf{A}$. *(1 mark)*

(b) State the invariant line of the transformation. *(1 mark)*

(c) Give a full geometrical description of this transformation. *(3 marks)*

3 Express the determinant $\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$ as the product of four linear factors. *(6 marks)*

4 The plane transformation T maps points (x, y) to points (x', y') such that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{where } \mathbf{A} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

(a) (i) State the line of invariant points of T. *(1 mark)*

(ii) Give a full geometrical description of T. *(2 marks)*

(b) Find $\mathbf{A}^2$, and hence give a full geometrical description of the single plane transformation given by the matrix $\mathbf{A}^2$. *(3 marks)*

5 A set of three planes is given by the system of equations

$$\begin{aligned} x + 3y - z &= 10 \\ 2x + ky + z &= -4 \\ 3x + 5y + (k-2)z &= k+4 \end{aligned}$$

where k is a constant.

(a) Show that $\begin{vmatrix} 1 & 3 & -1 \\ 2 & k & 1 \\ 3 & 5 & k-2 \end{vmatrix} = k^2 - 5k + 6$. *(2 marks)*

(b) In each of the following cases, determine the **number** of solutions of the given system of equations.

(i) $k = 1$.

(ii) $k = 2$.

(iii) $k = 3$. *(7 marks)*

(c) Give a geometrical interpretation of the significance of each of the three results in part (b) in relation to the three planes. *(3 marks)*

6 The matrices **P** and **Q** are given by

$$\mathbf{P} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & t & -2 \\ 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} 1 & 1 & 1 \\ -7 & -1 & 5 \\ 11 & -1 & -7 \end{bmatrix}$$

where t is a real constant.

(a) Find the value of t for which **P** is singular. *(2 marks)*

(b) (i) Determine the matrix $\mathbf{R} = \mathbf{PQ}$, giving its elements in terms of t where appropriate. *(3 marks)*

(ii) Find the value of t for which $\mathbf{R} = k\mathbf{I}$, for some integer k. *(2 marks)*

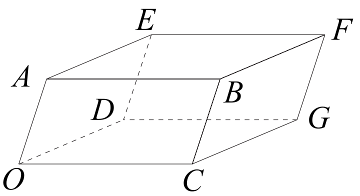
(iii) Hence find the matrix $\mathbf{Q}^{-1}$. *(1 mark)*

(c) In the case when $t = -3$, describe the geometrical transformation with matrix **R**. *(2 marks)*

Turn over for the next question

 Turn over ▶

7 The diagram shows the parallelepiped $OABCDEFG$.



Points A, B, C and D have position vectors

$$\mathbf{a} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ 1 \\ 6 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{d} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

respectively, relative to the origin O.

(a) Show that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are linearly dependent. *(1 mark)*

(b) Determine the volume of the parallelepiped. *(3 marks)*

(c) Determine a vector equation for the plane $ABDG$:

(i) in the form $\mathbf{r} = \mathbf{u} + \lambda\mathbf{v} + \mu\mathbf{w}$; *(2 marks)*

(ii) in the form $\mathbf{r} \,.\, \mathbf{n} = d$. *(4 marks)*

(d) Find cartesian equations for the line OF, and hence find the direction cosines of this line. *(4 marks)*

8 For real numbers a and b, with $b \neq 0$ and $b \neq \pm a$, the matrix

$$\mathbf{M} = \begin{bmatrix} a & b+a \\ b-a & -a \end{bmatrix}$$

(a) (i) Show that the eigenvalues of $\mathbf{M}$ are b and $-b$. *(3 marks)*

(ii) Show that $\begin{bmatrix} b+a \\ b-a \end{bmatrix}$ is an eigenvector of $\mathbf{M}$ with eigenvalue b. *(2 marks)*

(iii) Find an eigenvector of $\mathbf{M}$ corresponding to the eigenvalue $-b$. *(2 marks)*

(b) By writing $\mathbf{M}$ in the form $\mathbf{UDU}^{-1}$, for some suitably chosen diagonal matrix $\mathbf{D}$ and corresponding matrix $\mathbf{U}$, show that

$$\mathbf{M}^{11} = b^{10}\mathbf{M}$$ *(7 marks)*

END OF QUESTIONS

Answer **all** questions.

1 Show that the system of equations

$$\begin{aligned} x + 2y - z &= 0 \\ 3x - y + 4z &= 7 \\ 8x + y + 7z &= 30 \end{aligned}$$

is inconsistent. *(4 marks)*

2 (a) Show that $(a - b)$ is a factor of the determinant

$$\Delta = \begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ bc & ca & ab \end{vmatrix}$$

(2 marks)

(b) Factorise Δ completely into linear factors. *(5 marks)*

3 The points P, Q and R have position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively relative to an origin O, where

$$\mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} -3 \\ 4 \\ 20 \end{bmatrix} \text{ and } \mathbf{r} = \begin{bmatrix} 9 \\ 2 \\ 4 \end{bmatrix}$$

(a) (i) Determine $\mathbf{p} \times \mathbf{q}$. *(2 marks)*

(ii) Find the area of triangle OPQ. *(3 marks)*

(b) Use the scalar triple product to show that $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are linearly dependent, and interpret this result geometrically. *(3 marks)*

4 The matrices $\mathbf{M}_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ and $\mathbf{M}_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ represent the transformations A and B respectively.

(a) Give a full geometrical description of each of A and B. *(5 marks)*

(b) Transformation C is obtained by carrying out A followed by B.

(i) Find $\mathbf{M}_C$, the matrix of C. *(2 marks)*

(ii) Hence give a full geometrical description of the single transformation C. *(2 marks)*

5 (a) Find, to the nearest 0.1°, the acute angle between the planes with equations

$$\mathbf{r} \cdot (3\mathbf{i} - 4\mathbf{j} + \mathbf{k}) = 2 \quad \text{and} \quad \mathbf{r} \cdot (2\mathbf{i} + 12\mathbf{j} - \mathbf{k}) = 38$$ *(4 marks)*

(b) Write down cartesian equations for these two planes. *(2 marks)*

(c) (i) Find, in the form $\dfrac{x-a}{l} = \dfrac{y-b}{m} = \dfrac{z-c}{n}$, cartesian equations for the line of intersection of the two planes. *(5 marks)*

(ii) Determine the direction cosines of this line. *(2 marks)*

6 (a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$$ *(6 marks)*

(b) (i) Write down a diagonal matrix $\mathbf{D}$, and a suitable matrix $\mathbf{U}$, such that

$$\mathbf{X} = \mathbf{UDU}^{-1}$$ *(2 marks)*

(ii) Write down also the matrix $\mathbf{U}^{-1}$. *(1 mark)*

(iii) Use your results from parts (b)(i) and (b)(ii) to determine the matrix $\mathbf{X}^5$ in the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a, b, c and d are integers. *(3 marks)*

Turn over for the next question

7 The transformation S is a shear with matrix $\mathbf{M} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$. Points (x, y) are mapped under S to image points (x', y') such that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

(a) Find the equation of the line of invariant points of S. *(2 marks)*

(b) Show that all lines of the form $y = x + c$, where c is a constant, are invariant lines of S. *(3 marks)*

(c) Evaluate $\det \mathbf{M}$, and state the property of shears which is indicated by this result. *(2 marks)*

(d) Calculate, to the nearest degree, the acute angle between the line $y = -x$ and its image under S. *(3 marks)*

8 The matrix $\mathbf{P} = \begin{bmatrix} 4 & -1 & 2 \\ 1 & 1 & 3 \\ -2 & 0 & a \end{bmatrix}$, where a is constant.

(a) (i) Determine $\det \mathbf{P}$ as a linear expression in a. *(2 marks)*

(ii) Evaluate $\det \mathbf{P}$ in the case when $a = 3$. *(1 mark)*

(iii) Find the value of a for which $\mathbf{P}$ is singular. *(2 marks)*

(b) The 3×3 matrix $\mathbf{Q}$ is such that $\mathbf{PQ} = 25\mathbf{I}$.

Without finding **Q**:

(i) write down an expression for $\mathbf{P}^{-1}$ in terms of $\mathbf{Q}$; *(1 mark)*

(ii) find the value of the constant k such that $(\mathbf{PQ})^{-1} = k\mathbf{I}$; *(2 marks)*

(iii) determine the numerical value of $\det \mathbf{Q}$ in the case when $a = 3$. *(4 marks)*

END OF QUESTIONS

Answer **all** questions.

1 Given that $\mathbf{a} \times \mathbf{b} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ and that $\mathbf{a} \times \mathbf{c} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, determine:

(a) $\mathbf{c} \times \mathbf{a}$; *(1 mark)*

(b) $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$; *(2 marks)*

(c) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c})$; *(2 marks)*

(d) $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{c})$. *(1 mark)*

2 Factorise completely the determinant $\begin{vmatrix} y & x & x+y-1 \\ x & y & 1 \\ y+1 & x+1 & 2 \end{vmatrix}$. *(6 marks)*

3 Three points, A, B and C, have position vectors

$$\mathbf{a} = \begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

respectively.

(a) Using the scalar triple product, or otherwise, show that **a**, **b** and **c** are coplanar. *(2 marks)*

(b) (i) Calculate $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$. *(3 marks)*

(ii) Hence find, to three significant figures, the area of the triangle ABC . *(3 marks)*

4 Consider the following system of equations, where k is a real constant:

$$\begin{aligned} kx &+ 2y &+ z &= 5 \\ x &+ (k+1)y &- 2z &= 3 \\ 2x &- ky &+ 3z &= -11 \end{aligned}$$

(a) Show that the system does not have a unique solution when $k^2 = 16$. *(3 marks)*

(b) In the case when $k = 4$, show that the system is inconsistent. *(4 marks)*

(c) In the case when $k = -4$:

(i) solve the system of equations; *(5 marks)*

(ii) interpret this result geometrically. *(1 mark)*

5 The line l has equation $\mathbf{r} = \begin{bmatrix} 3 \\ 26 \\ -15 \end{bmatrix} + \lambda \begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix}$.

(a) Show that the point $P(-29, 42, -19)$ lies on l. *(1 mark)*

(b) Find:

(i) the direction cosines of l; *(2 marks)*

(ii) the acute angle between l and the z-axis. *(1 mark)*

(c) The plane Π has cartesian equation $3x - 4y + 5z = 100$.

(i) Write down a normal vector to Π. *(1 mark)*

(ii) Find the acute angle between l and this normal vector. *(4 marks)*

(d) Find the position vector of the point Q where l meets Π. *(4 marks)*

(e) Determine the shortest distance from P to Π. *(3 marks)*

Turn over for the next question

6 The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & t \end{bmatrix}$$

(a) Find, in terms of t, the matrices:

(i) **AB** ; *(3 marks)*

(ii) **BA** . *(2 marks)*

(b) Explain why **AB** is singular for all values of t. *(1 mark)*

(c) In the case when $t = -2$, show that the transformation with matrix **BA** is the combination of an enlargement, E, and a second transformation, F. Find the scale factor of E and give a full geometrical description of F. *(6 marks)*

7 (a) The matrix $\mathbf{M} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$ represents a shear.

(i) Find $\det \mathbf{M}$ and give a geometrical interpretation of this result. *(2 marks)*

(ii) Show that the characteristic equation of **M** is $\lambda^2 - 2\lambda + 1 = 0$, where λ is an eigenvalue of **M**. *(2 marks)*

(iii) Hence find an eigenvector of **M**. *(3 marks)*

(iv) Write down the equation of the line of invariant points of the shear. *(1 mark)*

(b) The matrix $\mathbf{S} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represents a shear.

(i) Write down the characteristic equation of **S**, giving the coefficients in terms of a, b, c and d. *(2 marks)*

(ii) State the numerical value of $\det \mathbf{S}$ and hence write down an equation relating a, b, c and d. *(2 marks)*

(iii) Given that the only eigenvalue of **S** is 1, find the value of $a + d$. *(2 marks)*

END OF QUESTIONS

Answer **all** questions.

1 Give a full geometrical description of the transformation represented by each of the following matrices:

(a) $\begin{bmatrix} 0.8 & 0 & -0.6 \\ 0 & 1 & 0 \\ 0.6 & 0 & 0.8 \end{bmatrix}$; *(3 marks)*

(b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. *(2 marks)*

2 It is given that $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$ and $\mathbf{c} = \mathbf{i} + 4\mathbf{j} + 28\mathbf{k}$.

(a) Determine:

(i) $\mathbf{a} \cdot \mathbf{b}$; *(1 mark)*

(ii) $\mathbf{a} \times \mathbf{b}$; *(2 marks)*

(iii) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. *(2 marks)*

(b) Describe the geometrical relationship between the vectors:

(i) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$; *(1 mark)*

(ii) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. *(1 mark)*

3 A shear S is represented by the matrix $\mathbf{A} = \begin{bmatrix} p & q \\ -q & r \end{bmatrix}$, where p, q and r are constants.

(a) By considering one of the geometrical properties of a shear, explain why $pr + q^2 = 1$. *(2 marks)*

(b) Given that $p = 4$ and that the image of the point $(-1, 2)$ under S is $(2, -1)$, find:

(i) the value of q and the value of r ; *(3 marks)*

(ii) the equation of the line of invariant points of S. *(3 marks)*

4 The matrix $\mathbf{T}$ has eigenvalues 2 and -2, with corresponding eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\3\end{bmatrix}$ respectively.

(a) Given that $\mathbf{T} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}$, where $\mathbf{D}$ is a diagonal matrix, write down suitable matrices $\mathbf{U}$, $\mathbf{D}$ and $\mathbf{U}^{-1}$. *(3 marks)*

(b) Hence prove that, for all **even** positive integers n,

$$\mathbf{T}^n = \mathrm{f}(n)\,\mathbf{I}$$

where $\mathrm{f}(n)$ is a function of n, and $\mathbf{I}$ is the 2×2 identity matrix. *(5 marks)*

5 A system of equations is given by

$$\begin{aligned} x + 3y + 5z &= -2 \\ 3x - 4y + 2z &= 7 \\ ax + 11y + 13z &= b \end{aligned}$$

where a and b are constants.

(a) Find the unique solution of the system in the case when $a = 3$ and $b = 2$. *(5 marks)*

(b) (i) Determine the value of a for which the system does not have a unique solution. *(3 marks)*

(ii) For this value of a, find the value of b such that the system of equations is consistent. *(4 marks)*

Turn over for the next question

6 (a) The line l has equation $\mathbf{r} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}$.

(i) Write down a vector equation for l in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$. *(1 mark)*

(ii) Write down cartesian equations for l. *(2 marks)*

(iii) Find the direction cosines of l and explain, geometrically, what these represent. *(3 marks)*

(b) The plane Π has equation $\mathbf{r} = \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

(i) Find an equation for Π in the form $\mathbf{r} \cdot \mathbf{n} = d$. *(4 marks)*

(ii) State the geometrical significance of the value of d in this case. *(1 mark)*

(c) Determine, to the nearest 0.1°, the angle between l and Π. *(4 marks)*

7 The non-singular matrix $\mathbf{M} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

(a) (i) Show that

$$\mathbf{M}^2 + 2\mathbf{I} = k\mathbf{M}$$

for some integer k to be determined. *(3 marks)*

(ii) By multiplying the equation in part (a)(i) by $\mathbf{M}^{-1}$, show that

$$\mathbf{M}^{-1} = a\mathbf{M} + b\mathbf{I}$$

for constants a and b to be found. *(3 marks)*

(b) (i) Determine the characteristic equation of $\mathbf{M}$ and show that $\mathbf{M}$ has a repeated eigenvalue, 1, and another eigenvalue, 2. *(6 marks)*

(ii) Give a full set of eigenvectors for each of these eigenvalues. *(5 marks)*

(iii) State the geometrical significance of each set of eigenvectors in relation to the transformation with matrix $\mathbf{M}$. *(3 marks)*

END OF QUESTIONS