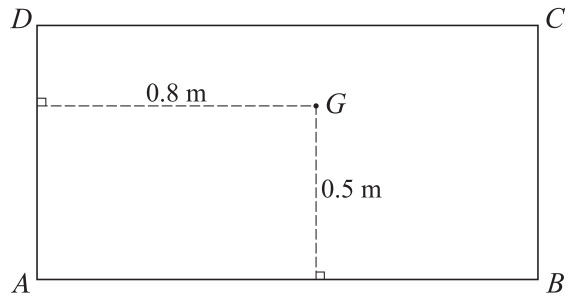
1 The driveshaft of an electric motor begins to rotate from rest and has constant angular acceleration. In the first 8 seconds it turns through 56 radians.

(i) Find the angular acceleration. [2]

(ii) Find the angle through which the driveshaft turns while its angular speed increases from $20\,\text{rad s}^{-1}$ to $36\,\text{rad s}^{-1}$. [2]

2 The region R is bounded by the curve $y = \sqrt{4a^2 - x^2}$ for $0 \leqslant x \leqslant a$, the x-axis, the y-axis and the line $x = a$, where a is a positive constant. The region R is rotated through 2π radians about the x-axis to form a uniform solid of revolution. Find the x-coordinate of the centre of mass of this solid. [7]

3



A non-uniform rectangular lamina $ABCD$ has mass 6 kg. The centre of mass G of the lamina is 0.8 m from the side AD and 0.5 m from the side AB (see diagram). The moment of inertia of the lamina about AD is $6.2\,\text{kg m}^2$ and the moment of inertia of the lamina about AB is $2.8\,\text{kg m}^2$.

The lamina rotates in a vertical plane about a fixed horizontal axis which passes through A and is perpendicular to the lamina.

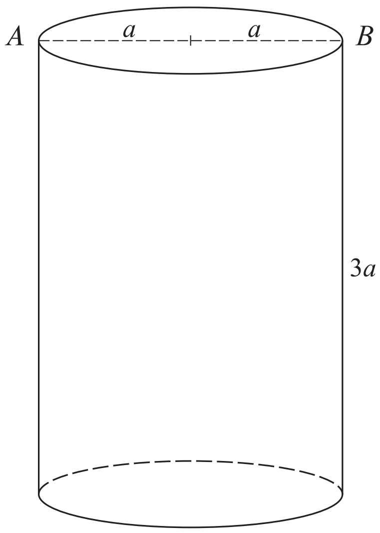
(i) Write down the moment of inertia of the lamina about this axis. [1]

The lamina is released from rest in the position where AB and DC are horizontal and DC is above AB. A frictional couple of constant moment opposes the motion. When AB is first vertical, the angular speed of the lamina is $2.4\,\text{rad s}^{-1}$.

(ii) Find the moment of the frictional couple. [5]

(iii) Find the angular acceleration of the lamina immediately after it is released. [3]

4



A uniform solid cylinder has radius a, height $3a$, and mass M. The line AB is a diameter of one of the end faces of the cylinder (see diagram).

(i) Show by integration that the moment of inertia of the cylinder about AB is $\frac{13}{4}Ma^2$. (You may assume that the moment of inertia of a uniform disc of mass m and radius a about a diameter is $\frac{1}{4}ma^2$.) [7]

The line AB is now fixed in a horizontal position and the cylinder rotates freely about AB, making small oscillations as a compound pendulum.

(ii) Find the approximate period of these small oscillations, in terms of a and g. [3]

5 A ship S is travelling with constant speed $12\,\text{m}\,\text{s}^{-1}$ on a course with bearing 345°. A patrol boat B spots the ship S when S is 2400 m from B on a bearing of 050°. The boat B sets off in pursuit, travelling with constant speed $v\,\text{m}\,\text{s}^{-1}$ in a straight line.

(i) Given that $v = 16$, find the bearing of the course which B should take in order to intercept S, and the time taken to make the interception. [8]

(ii) Given instead that $v = 10$, find the bearing of the course which B should take in order to get as close as possible to S. [4]

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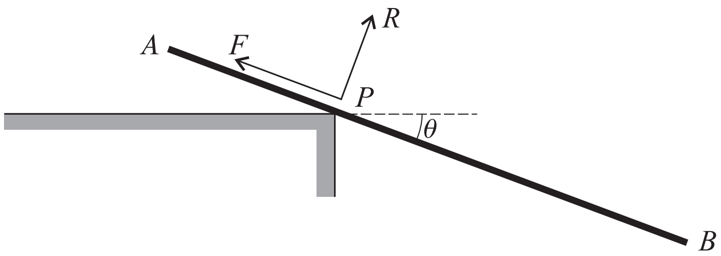


A uniform rod AB has mass m and length $2a$. The point P on the rod is such that $AP = \frac{2}{3}a$. The rod is placed in a horizontal position perpendicular to the edge of a rough horizontal table, with AP in contact with the table and PB overhanging the edge. The rod is released from rest in this position. When it has rotated through an angle θ, and no slipping has occurred at P, the normal reaction acting on the rod at P is R and the frictional force is F (see diagram).

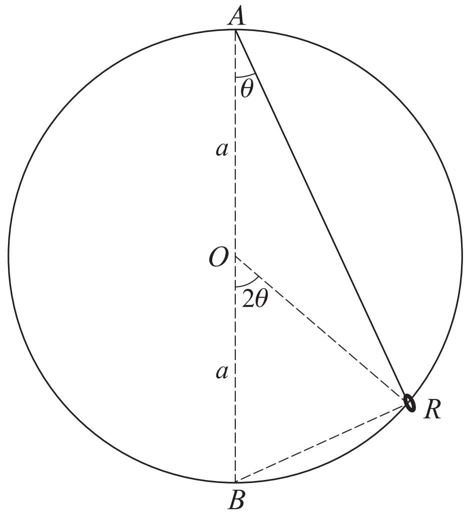
(i) Show that the angular acceleration of the rod is $\dfrac{3g\cos\theta}{4a}$. [4]

(ii) Find the angular speed of the rod, in terms of a, g and θ. [3]

(iii) Find F and R in terms of m, g and θ. [6]

(iv) Given that the coefficient of friction between the rod and the edge of the table is μ, show that the rod is on the point of slipping at P when $\tan\theta = \frac{1}{2}\mu$. [2]

7



A smooth circular wire, with centre O and radius a, is fixed in a vertical plane. The highest point on the wire is A and the lowest point on the wire is B. A small ring R of mass m moves freely along the wire. A light elastic string, with natural length a and modulus of elasticity $\frac{1}{2}mg$, has one end attached to A and the other end attached to R. The string AR makes an angle θ (measured anticlockwise) with the downward vertical, so that OR makes an angle 2θ with the downward vertical (see diagram). You may assume that the string does not become slack.

(i) Taking A as the level for zero gravitational potential energy, show that the total potential energy V of the system is given by

$$V = mga\left(\tfrac{1}{4} - \cos\theta - \cos^2\theta\right). \qquad [4]$$

(ii) Show that $\theta = 0$ is the only position of equilibrium. [3]

(iii) By differentiating the energy equation with respect to time t, show that

$$\frac{d^2\theta}{dt^2} = -\frac{g}{4a}\sin\theta(1 + 2\cos\theta). \qquad [5]$$

(iv) Deduce the approximate period of small oscillations about the equilibrium position $\theta = 0$. [3]

1 Two flywheels F and G are rotating freely, about the same axis and in the same direction, with angular speeds $21\,\text{rad}\,\text{s}^{-1}$ and $36\,\text{rad}\,\text{s}^{-1}$ respectively. The flywheels come into contact briefly, and immediately afterwards the angular speeds of F and G are $28\,\text{rad}\,\text{s}^{-1}$ and $34\,\text{rad}\,\text{s}^{-1}$, respectively, in the same direction. Given that the moment of inertia of F about the axis is $1.5\,\text{kg}\,\text{m}^2$, find the moment of inertia of G about the axis. [4]

2 A rotating turntable is slowing down with constant angular deceleration. It makes 16 revolutions as its angular speed decreases from $8\,\text{rad}\,\text{s}^{-1}$ to rest.

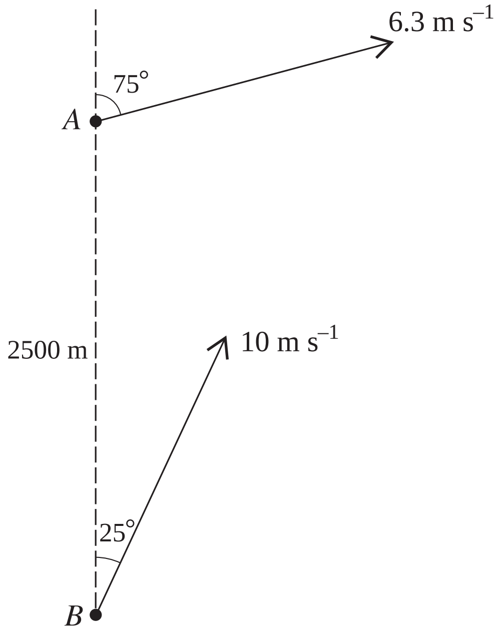
(i) Find the angular deceleration of the turntable. [2]

(ii) Find the angular speed of the turntable at the start of its last complete revolution before coming to rest. [2]

(iii) Find the time taken for the turntable to make its last complete revolution before coming to rest. [2]

3 The region bounded by the curve $y = 2x + x^2$ for $0 \leqslant x \leqslant 3$, the x-axis, and the line $x = 3$, is occupied by a uniform lamina. Find the coordinates of the centre of mass of this lamina. [9]

4



A boat A is travelling with constant speed $6.3\,\text{m}\,\text{s}^{-1}$ on a course with bearing 075°. Boat B is travelling with constant speed $10\,\text{m}\,\text{s}^{-1}$ on a course with bearing 025°. At one instant, A is 2500 m due north of B (see diagram).

(i) Find the magnitude and bearing of the velocity of A relative to B. [5]

(ii) Find the shortest distance between A and B in the subsequent motion. [3]

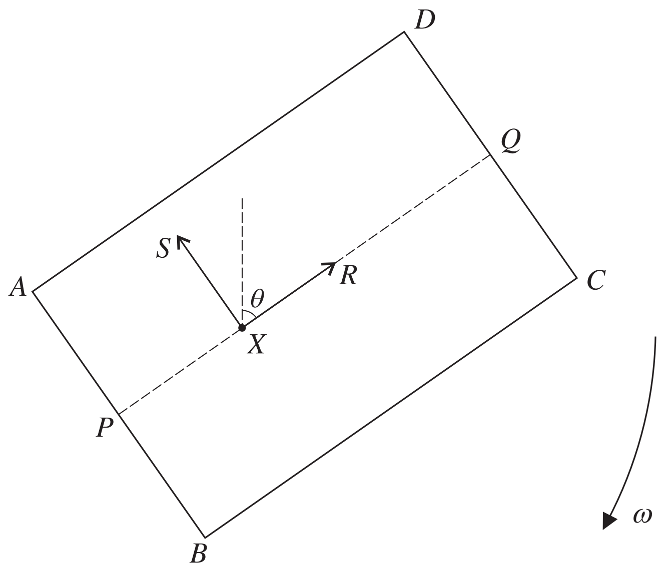
5 The region bounded by the curve $y = \sqrt{ax}$ for $a \leqslant x \leqslant 4a$ (where a is a positive constant), the x-axis, and the lines $x = a$ and $x = 4a$, is rotated through 2π radians about the x-axis to form a uniform solid of revolution of mass m.

(i) Show that the moment of inertia of this solid about the x-axis is $\frac{7}{5}ma^2$. [8]

The solid is free to rotate about a fixed horizontal axis along the line $y = a$, and makes small oscillations as a compound pendulum.

(ii) Find, in terms of a and g, the approximate period of these small oscillations. [4]

6



A uniform rectangular lamina $ABCD$ has mass m and sides $AB = 2a$ and $BC = 3a$. The mid-point of AB is P and the mid-point of CD is Q. The lamina is rotating freely in a vertical plane about a fixed horizontal axis which is perpendicular to the lamina and passes through the point X on PQ where $PX = a$. Air resistance may be neglected. When Q is vertically above X, the angular speed is $\sqrt{\frac{9g}{10a}}$. When XQ makes an angle θ with the upward vertical, the angular speed is ω, and the force acting on the lamina at X has components R parallel to PQ and S parallel to BA (see diagram).

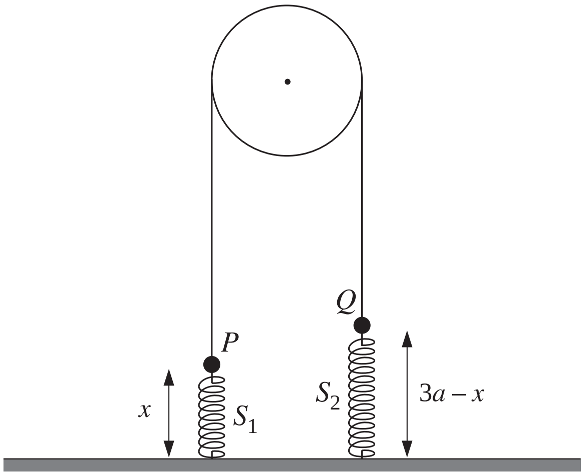
(i) Show that the moment of inertia of the lamina about the axis through X is $\frac{4}{3}ma^2$. [3]

(ii) At an instant when $\cos\theta = \frac{3}{5}$, show that $\omega^2 = \frac{6g}{5a}$. [3]

(iii) At an instant when $\cos\theta = \frac{3}{5}$, show that $R = 0$, and given also that $\sin\theta = \frac{4}{5}$ find S in terms of m and g. [9]

[Turn over

7



Particles P and Q, with masses $3m$ and $2m$ respectively, are connected by a light inextensible string passing over a smooth light pulley. The particle P is connected to the floor by a light spring S_1 with natural length a and modulus of elasticity mg. The particle Q is connected to the floor by a light spring S_2 with natural length a and modulus of elasticity $2mg$. The sections of the string not in contact with the pulley, and the two springs, are vertical. Air resistance may be neglected. The particles P and Q move vertically and the string remains taut; when the length of S_1 is x, the length of S_2 is $(3a - x)$ (see diagram).

(i) Find the total potential energy of the system (taking the floor as the reference level for gravitational potential energy). Hence show that $x = \frac{4}{3}a$ is a position of stable equilibrium. [9]

(ii) By differentiating the energy equation, and substituting $x = \frac{4}{3}a + y$, show that the motion is simple harmonic, and find the period. [9]

1 A wheel is rotating and is slowing down with constant angular deceleration. The initial angular speed is $80\,\text{rad}\,\text{s}^{-1}$, and after 15 s the wheel has turned through 1020 radians.

(i) Find the angular deceleration of the wheel. **[2]**

(ii) Find the angle through which the wheel turns in the last 5 s before it comes to rest. **[2]**

(iii) Find the total number of revolutions made by the wheel from the start until it comes to rest. **[3]**

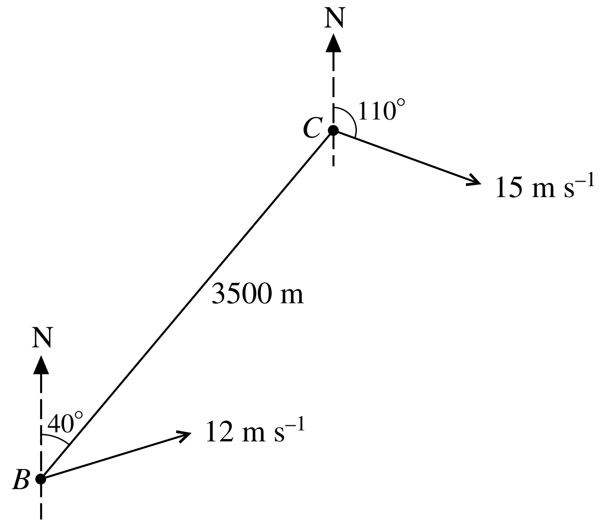
2 The region bounded by the x-axis, the y-axis, the line $x = \ln 3$, and the curve $y = e^{-x}$ for $0 \leqslant x \leqslant \ln 3$, is occupied by a uniform lamina. Find, in an exact form, the coordinates of the centre of mass of this lamina. **[9]**

3 A circular disc is rotating in a horizontal plane with angular speed $16\,\text{rad}\,\text{s}^{-1}$ about a fixed vertical axis passing through its centre O. The moment of inertia of the disc about the axis is $0.9\,\text{kg}\,\text{m}^2$. A particle, initially at rest just above the surface of the disc, drops onto the disc and sticks to it at a point 0.4 m from O. Afterwards, the angular speed of the disc with the particle attached is $15\,\text{rad}\,\text{s}^{-1}$.

(i) Find the mass of the particle. **[4]**

(ii) Find the loss of kinetic energy. **[3]**

4



From a boat B, a cruiser C is observed 3500 m away on a bearing of 040°. The cruiser C is travelling with constant speed $15\,\text{m}\,\text{s}^{-1}$ along a straight line course with bearing 110° (see diagram). The boat B travels with constant speed $12\,\text{m}\,\text{s}^{-1}$ on a straight line course which takes it as close as possible to the cruiser C.

(i) Show that the bearing of the course of B is 073°, correct to the nearest degree. **[4]**

(ii) Find the magnitude and the bearing of the velocity of C relative to B. **[3]**

(iii) Find the shortest distance between B and C in the subsequent motion. **[3]**

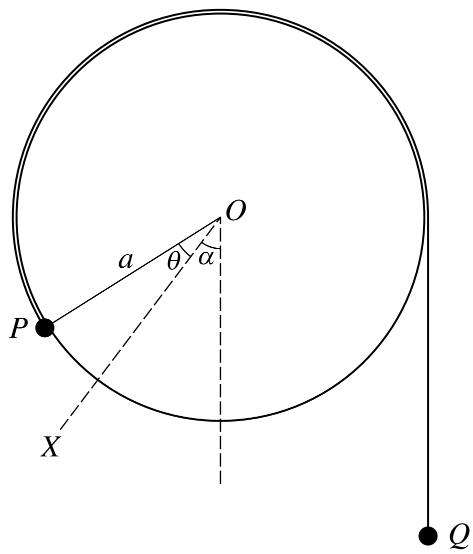
5 A uniform rod AB has mass m and length $6a$. The point C on the rod is such that $AC = a$. The rod can rotate freely in a vertical plane about a fixed horizontal axis passing through C and perpendicular to the rod.

(i) Show by integration that the moment of inertia of the rod about this axis is $7ma^2$. **[5]**

The rod starts at rest with B vertically below C. A couple of constant moment $\dfrac{6mga}{\pi}$ is then applied to the rod.

(ii) Find, in terms of a and g, the angular speed of the rod when it has turned through one and a half revolutions. **[6]**

6



A light pulley of radius a is free to rotate in a vertical plane about a fixed horizontal axis passing through its centre O. Two particles, P of mass $5m$ and Q of mass $3m$, are connected by a light inextensible string. The particle P is attached to the circumference of the pulley, the string passes over the top of the pulley, and Q hangs below the pulley on the opposite side to P. The section of string not in contact with the pulley is vertical. The fixed line OX makes an angle α with the downward vertical, where $\cos\alpha = \frac{4}{5}$, and OP makes an angle θ with OX (see diagram).

You are given that the total potential energy of the system (using a suitable reference level) is V, where

$$V = mga(3\sin\theta - 4\cos\theta - 3\theta).$$

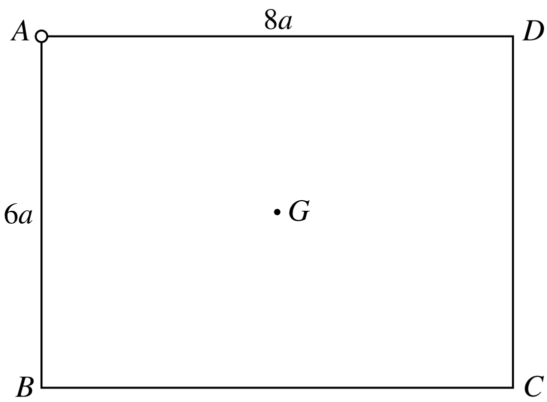
(i) Show that $\theta = 0$ is a position of stable equilibrium. **[5]**

(ii) Show that the kinetic energy of the system is $4ma^2\dot{\theta}^2$. **[2]**

(iii) By differentiating the energy equation, then making suitable approximations for $\sin\theta$ and $\cos\theta$, find the approximate period of small oscillations about the equilibrium position $\theta = 0$. **[5]**

[Question 7 is printed overleaf.]

7



The diagram shows a uniform rectangular lamina $ABCD$ with $AB = 6a$, $AD = 8a$ and centre G. The mass of the lamina is m. The lamina rotates freely in a vertical plane about a fixed horizontal axis passing through A and perpendicular to the lamina.

(i) Find the moment of inertia of the lamina about this axis. **[3]**

The lamina is released from rest with AD horizontal and BC below AD.

(ii) For an instant during the subsequent motion when AD is vertical, show that the angular speed of the lamina is $\sqrt{\frac{3g}{50a}}$ and find its angular acceleration. **[5]**

At an instant when AD is vertical, the force acting on the lamina at A has magnitude F.

(iii) By finding components parallel and perpendicular to GA, or otherwise, show that $F = \frac{\sqrt{493}}{20}mg$. **[8]**

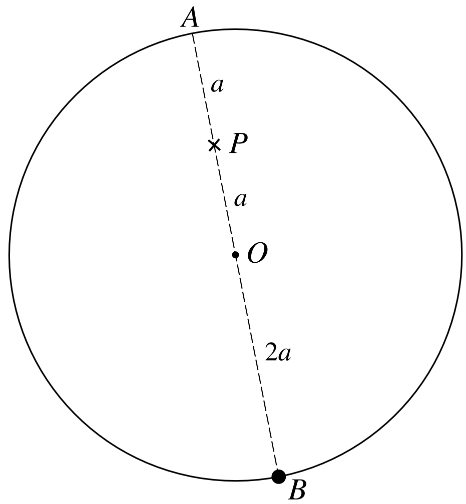
1 A top is set spinning with initial angular speed 83 rad s^{-1}, and it slows down with constant angular deceleration. When it has turned through 1000 radians, its angular speed is 67 rad s^{-1}.

(i) Find the angular deceleration of the top. **[2]**

(ii) Find the time taken, from the start, for the top to turn through 400 radians. **[4]**

2 The region R is bounded by the x-axis, the lines $x = a$ and $x = 2a$, and the curve $y = \frac{a^3}{x^2}$ for $a \leqslant x \leqslant 2a$, where a is a positive constant. A uniform solid of revolution is formed by rotating R through 2π radians about the x-axis. Find the x-coordinate of the centre of mass of this solid. **[7]**

3



A uniform circular disc has mass $4m$, radius $2a$ and centre O. The points A and B are at opposite ends of a diameter of the disc, and the mid-point of OA is P. A particle of mass m is attached to the disc at B. The resulting compound pendulum is in a vertical plane and is free to rotate about a fixed horizontal axis passing through P and perpendicular to the disc (see diagram). The pendulum makes small oscillations.

(i) Find the moment of inertia of the pendulum about the axis. **[4]**

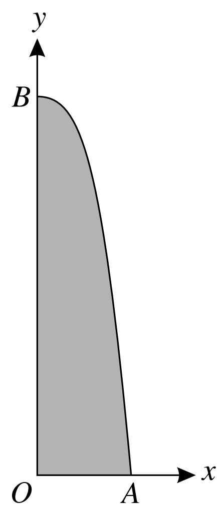
(ii) Find the approximate period of the small oscillations. **[4]**

4 From a helicopter, a small plane is spotted 3750 m away on a bearing of 075°. The plane is at the same altitude as the helicopter, and is flying with constant speed 62 m s^{-1} in a horizontal straight line on a bearing of 295°. The helicopter flies with constant speed 48 m s^{-1} in a straight line, and intercepts the plane.

(i) Find the bearings of the two possible directions in which the helicopter could fly. **[5]**

(ii) Given that interception occurs in the shorter of the two possible times, find the time taken to make the interception. **[4]**

5



A uniform lamina of mass 63 kg occupies the region bounded by the x-axis, the y-axis, and the curve $y = 8 - x^3$ for $0 \leqslant x \leqslant 2$. The unit of length is the metre. The vertices of the lamina are O (0, 0), A (2, 0) and B (0, 8) (see diagram).

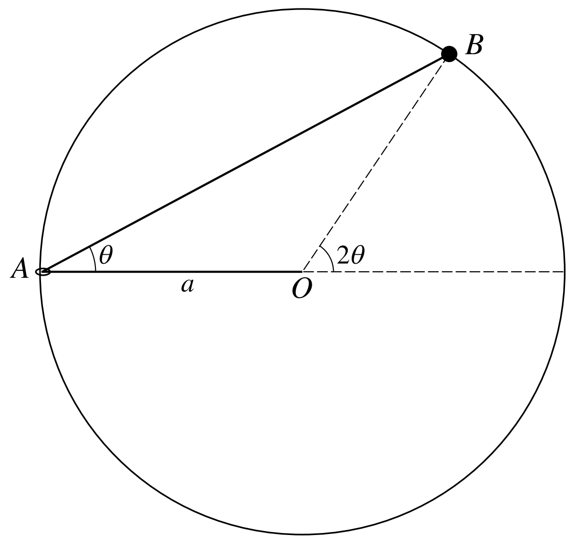
(i) Show that the moment of inertia of this lamina about OB is 56 kg m^2. **[6]**

It is given that the moment of inertia of the lamina about OA is 1036.8 kg m^2, and the centre of mass of the lamina has coordinates $\left(\frac{4}{5}, \frac{24}{7}\right)$. The lamina is free to rotate in a vertical plane about a fixed horizontal axis passing through O and perpendicular to the lamina. Starting with the lamina at rest with B vertically above O, a couple of constant anticlockwise moment 800 N m is applied to the lamina.

(ii) Show that the lamina begins to rotate anticlockwise. **[2]**

(iii) Find the angular speed of the lamina at the instant when OB first becomes horizontal. **[6]**

6



A smooth circular wire, with centre O and radius a, is fixed in a vertical plane, and the point A is on the wire at the same horizontal level as O. A small bead B of mass m can move freely on the wire. A light elastic string, with natural length a and modulus of elasticity $\sqrt{3}mg$, passes through a fixed ring at A, and has one end fixed at O and the other end attached to B. The section AB of the string is at an angle θ above the horizontal, where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, so that OB is at an angle 2θ to the horizontal (see diagram).

(i) Taking O as the reference level for gravitational potential energy, show that the total potential energy of the system is

$$mga(\sqrt{3} + \sqrt{3}\cos 2\theta + \sin 2\theta).$$ **[4]**

(ii) Find the two values of θ for which the system is in equilibrium. **[5]**

(iii) For each position of equilibrium, determine whether it is stable or unstable. **[4]**

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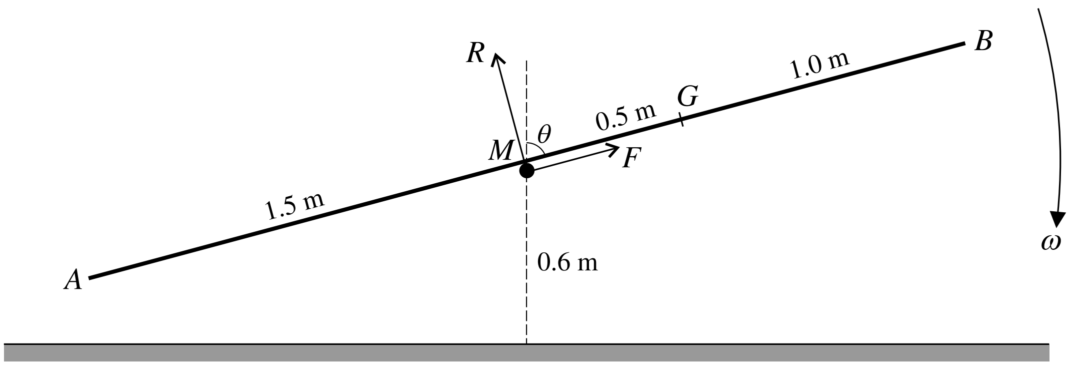


A thin horizontal rail is fixed at a height of 0.6 m above horizontal ground. A non-uniform straight rod AB has mass 6 kg and length 3 m; its centre of mass G is 2 m from A and 1 m from B, and its moment of inertia about a perpendicular axis through its mid-point M is $4.9\,\text{kg}\,\text{m}^2$. The rod is placed in a vertical plane perpendicular to the rail, with A on the ground and M in contact with the rail. It is released from rest in this position, and begins to rotate about M, without slipping on the rail. When the angle between AB and the upward vertical is θ radians, the rod has angular speed $\omega\,\text{rad}\,\text{s}^{-1}$, the frictional force in the direction AB is F N, and the normal reaction is R N (see diagram).

(i) Show that $\omega^2 = 4.8 - 12\cos\theta$. **[3]**

(ii) Find the angular acceleration of the rod in terms of θ. **[2]**

(iii) Show that $F = 94.8\cos\theta - 14.4$, and find R in terms of θ. **[6]**

(iv) Given that the coefficient of friction between the rod and the rail is 0.9, show that the rod will slip on the rail before B hits the ground. **[4]**

Answer **all** questions.

1 The time T taken for a simple pendulum to make a single small oscillation is thought to depend only on its length l, its mass m and the acceleration due to gravity g.

By using dimensional analysis:

(a) show that T does **not** depend on m; *(3 marks)*

(b) express T in terms of l, g and k, where k is a dimensionless constant. *(4 marks)*

2 Three smooth spheres A, B and C of equal radii and masses m, m and $2m$ respectively lie at rest on a smooth horizontal table. The centres of the spheres lie in a straight line with B between A and C. The coefficient of restitution between any two spheres is e.

The sphere A is projected directly towards B with speed u and collides with B.

(a) Find, in terms of u and e, the speed of B immediately after the impact between A and B. *(5 marks)*

(b) The sphere B subsequently collides with C. The speed of C immediately after this collision is $\frac{3}{8}u$. Find the value of e. *(7 marks)*

3 A ball of mass 0.45 kg is travelling horizontally with speed $15\,\text{m}\,\text{s}^{-1}$ when it strikes a fixed vertical bat directly and rebounds from it. The ball stays in contact with the bat for 0.1 seconds.

At time t seconds after first coming into contact with the bat, the force exerted on the ball by the bat is $1.4 \times 10^5(t^2 - 10t^3)$ newtons, where $0 \leqslant t \leqslant 0.1$.

In this simple model, ignore the weight of the ball and model the ball as a particle.

(a) Show that the magnitude of the impulse exerted by the bat on the ball is 11.7 N s, correct to three significant figures. *(4 marks)*

(b) Find, to two significant figures, the speed of the ball immediately after the impact. *(4 marks)*

(c) Give a reason why the speed of the ball immediately after the impact is different from the speed of the ball immediately before the impact. *(1 mark)*

4 The unit vectors **i** and **j** are directed due east and due north respectively.

Two cyclists, Aazar and Ben, are cycling on straight horizontal roads with constant velocities of $(6\mathbf{i} + 12\mathbf{j})\,\mathrm{km\,h^{-1}}$ and $(12\mathbf{i} - 8\mathbf{j})\,\mathrm{km\,h^{-1}}$ respectively. Initially, Aazar and Ben have position vectors $(5\mathbf{i} - \mathbf{j})$ km and $(18\mathbf{i} + 5\mathbf{j})$ km respectively, relative to a fixed origin.

(a) Find, as a vector in terms of **i** and **j**, the velocity of Ben relative to Aazar. *(2 marks)*

(b) The position vector of Ben relative to Aazar at time t hours after they start is **r** km.

Show that

$$\mathbf{r} = (13 + 6t)\mathbf{i} + (6 - 20t)\mathbf{j}$$ *(4 marks)*

(c) Find the value of t when Aazar and Ben are closest together. *(6 marks)*

(d) Find the closest distance between Aazar and Ben. *(2 marks)*

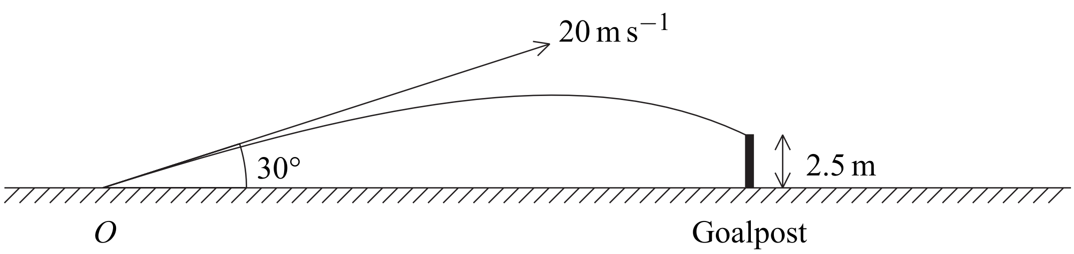
5 A football is kicked from a point O on a horizontal football ground with a velocity of $20\,\mathrm{m\,s^{-1}}$ at an angle of elevation of 30°. During the motion, the horizontal and upward vertical displacements of the football from O are x metres and y metres respectively.

(a) Show that x and y satisfy the equation

$$y = x\tan 30° - \frac{gx^2}{800\cos^2 30°}$$ *(6 marks)*

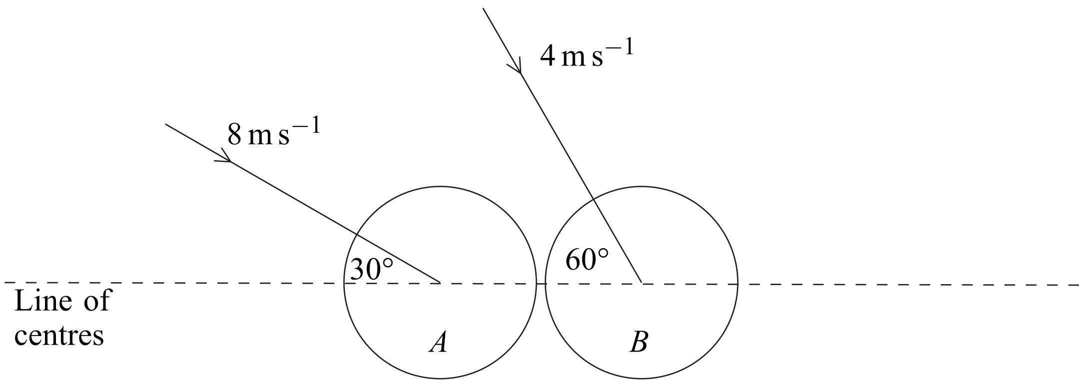
(b) On its downward flight the ball hits the horizontal crossbar of the goal at a point which is 2.5 m above the ground. Using the equation given in part (a), find the horizontal distance from O to the goal.

(4 marks)



(c) State **two** modelling assumptions that you have made. *(2 marks)*

6 Two smooth billiard balls A and B, of identical size and equal mass, move towards each other on a horizontal surface and collide. Just before the collision, A has velocity $8\,\text{m}\,\text{s}^{-1}$ in a direction inclined at 30° to the line of centres of the balls, and B has velocity $4\,\text{m}\,\text{s}^{-1}$ in a direction inclined at 60° to the line of centres, as shown in the diagram.



The coefficient of restitution between the balls is $\frac{1}{2}$.

(a) Find the speed of B immediately after the collision. *(9 marks)*

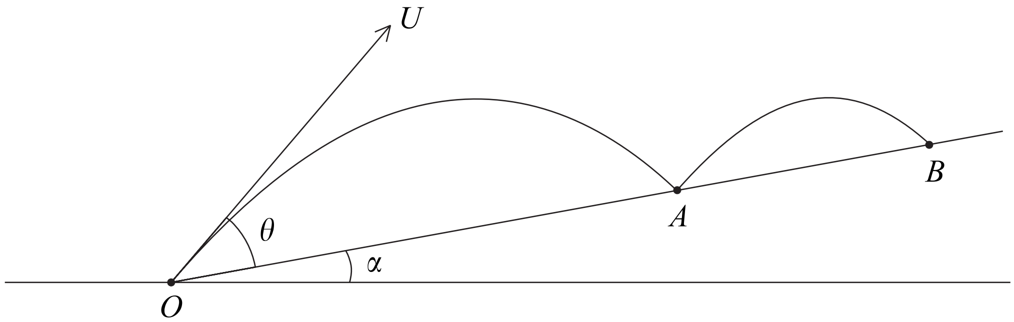
(b) Find the angle between the velocity of B and the line of centres of the balls immediately after the collision. *(2 marks)*

7 A projectile is fired from a point O on the slope of a hill which is inclined at an angle α to the horizontal. The projectile is fired up the hill with velocity U at an angle θ above the hill and first strikes it at a point A. The projectile is modelled as a particle and the hill is modelled as a plane with OA as a line of greatest slope.

(a) (i) Find, in terms of U, g, α and θ, the time taken by the projectile to travel from O to A. *(3 marks)*

(ii) Hence, or otherwise, show that the magnitude of the component of the velocity of the projectile perpendicular to the hill, when it strikes the hill at the point A, is the same as it was initially at O. *(3 marks)*

(b) The projectile rebounds and strikes the hill again at a point B. The hill is smooth and the coefficient of restitution between the projectile and the hill is e.



Find the ratio of the time of flight from O to A to the time of flight from A to B. Give your answer in its simplest form. *(4 marks)*

END OF QUESTIONS